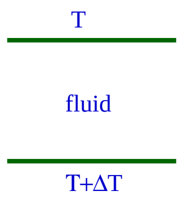
1. Why turbulence occur?

Hydrodynamic Instability

Rayleigh-Benard Instability:



- ❑ Driving force: buoyancy
- ❑ Damping force: viscous dissipation

Low ΔT: motionless, pure thermal conduction

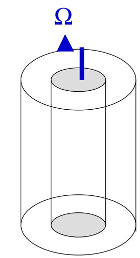
Higher ΔT: steady convection roll

Even Higher ΔT: unsteady, turbulent

stability of a heated plate

Hydrodynamic Instability

Centrifugal Instability:



- ❑ Driving force: centrifugal force
- ❑ Damping force: viscous dissipation

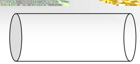
Low Ω: laminar, concentric streamlines

Higher Ω: steady convection roll

Even Higher Ω: unsteady, turbulent

Hydrodynamic Instability

Reynolds' experiment: circular pipe flow



Low Re: steady, axisymmetric, parallel, parabolic Poiseuille flow

Re >2000: sporadic bursts of turbulence alternate with laminar (intermittence)

High Re: fully turbulent

❖ Transition from laminar to turbulent flow need not be simply a function of the base flow considered, but also of the amplitude and type of perturbations.

❖Multiple paths to transition through different sequences of laminar flow instabilities are extremely possible.

Hopf bifurcation

dynamic system:

$$\frac{dx}{dt} = \mu x - y + 2\sigma x(x^2+y^2) - x(x^2+y^2)^2$$

$$\frac{dy}{dt} = x + \mu y + 2\sigma y(x^2+y^2) - y(x^2+y^2)^2$$

Use polar coordinates:

$$\frac{dr}{dt} = \mu r + 2\sigma r^3 - r^5$$

$$\frac{d\theta}{dt} = 1$$

fixed point : $(x, y) = (0,0)$

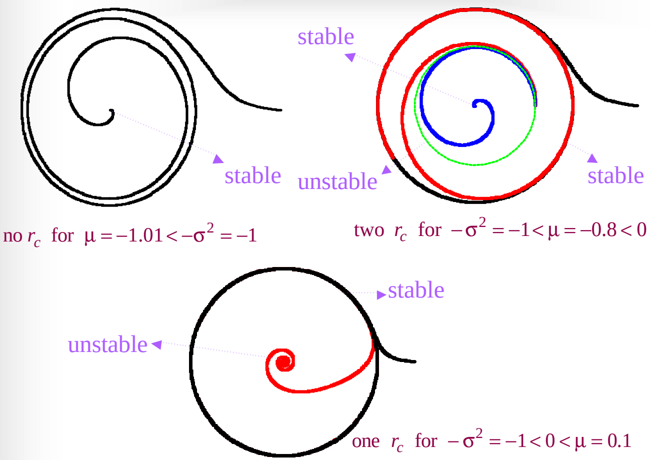
periodic orbit : $\frac{dr}{dt} = 0$ at $\mu + 2\sigma r_c^2 - r_c^4 = 0$

no such r_c if $\mu < -\sigma^2$

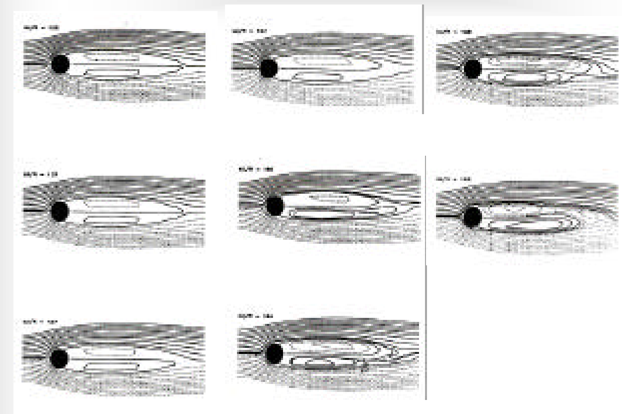
two r_c's if $-\sigma^2 < \mu < 0$

one r_c if $0 < \mu$

Hopf bifurcation

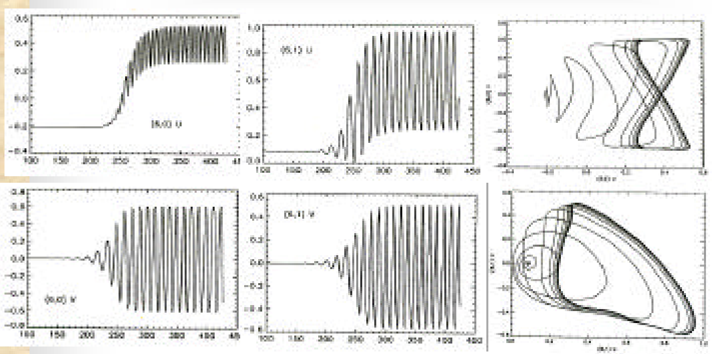


Hopf bifurcation



vortex stree behind a vertical plate

Hopf bifurcation



velocity signals

3. Vorticity and Vortex Stretching

Velocity v.s. Vorticity:

$$\vec{\omega} = \nabla \times \vec{u} \quad ; \quad \nabla^2 \vec{u} = -\nabla \times \vec{\omega} \ \text{(incompressible)}$$

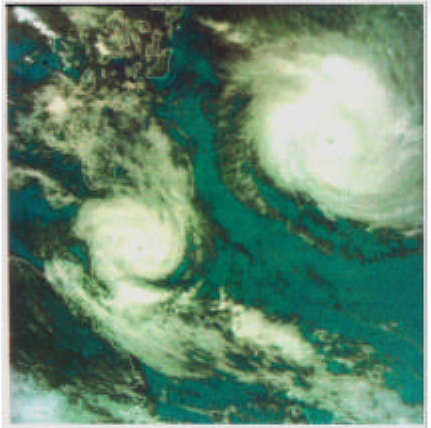
$$\nabla \times \vec{\omega} = \nabla \times \nabla \times \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$$

- Vorticity field is usually more compact than velocity field. It is therefore more economic to describe a flow in terms of its vorticity field.

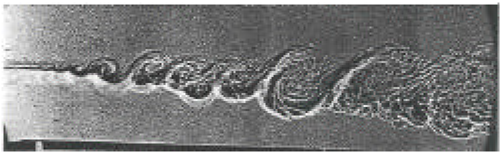
3. Vorticity and Vortex Stretching

landing

3. Vorticity and Vortex Stretching



3. Vorticity and Vortex Stretching



Mixing layer between helium and nitrogen

$U_2/U_1 = 0.38;\ \rho_2/\rho_1 = 7;\ \rho_1 U_1 L/\mu_1 = 6\times10^4$

(L is the width of the picture)

Tensor Notation

$$\vec{\omega} \Leftrightarrow \omega_i$$

$$\nabla\cdot\vec{u} \Leftrightarrow \sum_{j=1}^{3}\frac{\partial u_j}{\partial x_j} \Leftrightarrow \frac{\partial u_j}{\partial x_j}$$

$$\delta_{ij} = \begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise}\end{cases}$$

$$\delta_{ii} = \sum_{i=1}^{3}\delta_{ii} = \delta_{11}+\delta_{22}+\delta_{33} = 3$$

$$\varepsilon_{ijk} = \begin{cases}\pm1 & \text{if } i\neq j\neq k \quad (\varepsilon_{123}=+1)\\ 0 & \text{otherwise}\end{cases}$$

$$\varepsilon_{mij}\varepsilon_{mrs} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}$$

Tensor Notation

$$(\vec{a}\cdot\nabla)\vec{b} \Leftrightarrow \sum_{j=1}^{3} a_j \frac{\partial b_i}{\partial x_j} \Leftrightarrow a_j \frac{\partial b_i}{\partial x_j}$$

$$\vec{u}\times\vec{\omega} \Leftrightarrow \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} u_j \omega_k \Leftrightarrow \varepsilon_{ijk} u_j \omega_k$$

$$\nabla\times\vec{u} \Leftrightarrow \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \Leftrightarrow \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

3. Vorticity and Vortex Stretching

Physical Significance

1. related to the rotational tensor

Consider the relative velocity of two nearby fluid particles separated by a distance $\delta\vec{x}$:

$$\delta u_i = \delta x_j \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta x_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\delta x_j$$

$$\delta u_i = S_{ij}\delta x_j + \Omega_{ij}\delta x_j$$

S_{ij} = strain tensor (angular deformation; straining)

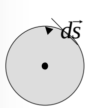
Ω_{ij} = rotational tensor

$$\omega_i = -\varepsilon_{ijk}\Omega_{jk} \quad ; \quad \Omega_{ij} = -\tfrac{1}{2}\varepsilon_{ijk}\omega_k$$

Physical Significance

2. Twice the average angular velocity

Consider an infinitesimal circle around a point in space:



$$\text{average angular velocity} = \frac{1}{2\pi r}\oint \frac{\vec{u}\cdot d\vec{s}}{r}$$

$$= \frac{1}{2\pi r}\iint_A (\nabla\times\vec{u})\cdot d\vec{A} = \frac{1}{2\pi r}\iint_A \vec{\omega}\cdot d\vec{A}$$

As $r \to 0$, $$= \frac{1}{2\pi r}\left(\omega_n \pi r^2\right) = \tfrac{1}{2}\omega_n$$

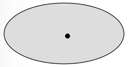
The average angular velocity around an infinitesimal circle =

half the vorticity component in the direction of the area enclosed by the circle.

Physical Significance

3. Proportional to the angular momentum of fluid particles whose inertia tensor has spherical symmetry

Consider a fluid region δV centered at $\vec{x}$:



angular momentum about its centroid :

$$\delta\vec{A} = \rho\iiint_{\delta V} \delta\vec{x}\times\delta\vec{u}\, dV$$

$$\delta u_i = S_{ij}\delta x_j + \Omega_{ij}\delta x_j$$

$$\Omega_{km} = -\tfrac{1}{2}\varepsilon_{kml}\omega_l$$

$$\delta\vec{A} = \varepsilon_{ijk} S_{km} I_{jm} + \tfrac{1}{2}\left(\delta_{ij} I_{mm} - I_{ij}\right)\omega_j$$

$$I_{jm} = \rho\iiint_{\delta V} \delta x_j \delta x_m\, dV = \text{inertia tensor}$$

If spherical symmetry $\left(I_{ij} = I\,\delta_{ij}\right)$, $\delta A_i = I\omega_i$ or $\delta\vec{A} = I\,\vec{\omega}$

3. Vorticity and Vortex Stretching

Mathematical Description

Incompressible Navier-Stokes flows of Newtonian fluid with constant properties:

$$\nabla \cdot \vec{u} = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\frac{1}{\rho}\nabla p + \vec{g} + \frac{\mu}{\rho}\nabla^2\vec{u}$$

$$\nabla \times \left\{ \frac{\partial \vec{u}}{\partial t} + \nabla\left(\tfrac{1}{2}\vec{u}\cdot\vec{u}\right) - \tfrac{1}{2}\vec{u}\times\vec{\omega} = -\frac{1}{\rho}\nabla p + \vec{g} + \frac{\mu}{\rho}\nabla^2\vec{u} \right\}$$

$$\underbrace{\frac{\partial \vec{\omega}}{\partial t} + (\vec{u}\cdot\nabla)\vec{\omega}} - (\vec{\omega}\cdot\nabla)\vec{u} = -\nabla\frac{1}{\rho}\times\nabla p + \nabla\times\vec{g} + \frac{\mu}{\rho}\nabla^2\vec{\omega}$$

$\frac{D\vec{\omega}}{Dt}$ = Lagrangian time change rate of vorticity

Mathematical Description

$$RHS = \underbrace{-\nabla\frac{1}{\rho}\times\nabla p} + \underbrace{\nabla\times\vec{g}} + \frac{\mu}{\rho}\nabla^2\vec{\omega}$$

⇐ viscous diffusion

pressure toque

body-force torque

nonzero when the geometric centroid is not the mass centroid

zero if barotropic, i.e. $\rho = \rho(p)$

zero if body force is irrotational or conservative, i.e. $\vec{g} = -\nabla U$, where U is some scalar function, called potential function

nonzero if rotational body force, e.g. Coriolis force

3. Vorticity and Vortex Stretching

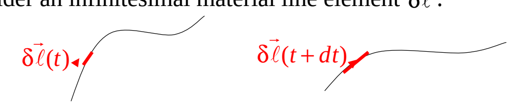
$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u}\cdot\nabla)\vec{\omega} - (\vec{\omega}\cdot\nabla)\vec{u} = -\nabla\frac{1}{\rho}\times\nabla p + \nabla\times\vec{g} + \frac{\mu}{\rho}\nabla^2\vec{\omega}$$

vortex stretching

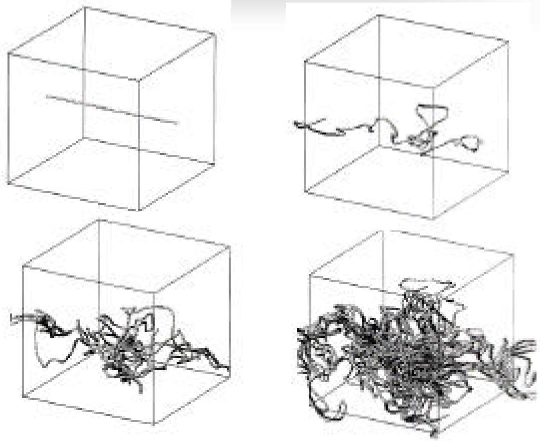
barotropic fluids, conservative body force, inviscid:

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{u}$$

Consider an infinitesimal material line element $\delta\vec{\ell}$:

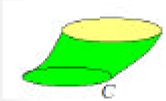


$\frac{D\delta\vec{\ell}}{Dt} = \delta\vec{u} = \left(\delta\vec{\ell}\cdot\nabla\right)\vec{u}$ = rate at which two initially infinitely close-by fluid particles separate



powerfulness of turbulent stretching

vorticity/material tube:



$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{u}$$

$$\frac{D}{Dt}\int_A \omega_i dA = \int_A \frac{D\omega_i}{Dt} dA = \int_A \omega_j \frac{\partial u_i}{\partial x_j} dA$$

$$= \int_A \frac{\partial}{\partial x_j}\left(u_i \omega_j\right) dA = \oint_C u_i \omega_j n_j ds = 0$$

stretched (thinning) ⇨ rotation speed up

compressed (flatten) ⇨ rotation slow down

barotropic fluids, conservative body force:

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{u} + \nu\nabla^2\vec{\omega}$$

stretching + diffusion

nonlinear + linear

magnitude:

$$\frac{D}{Dt}\left(\frac{1}{2}\omega_i\omega_i\right) = \omega_i\omega_j\frac{\partial u_i}{\partial x_j} + \nu\frac{\partial}{\partial x_j}\left(\omega_i\frac{\partial \omega_i}{\partial x_j}\right) - \nu\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j}$$

viscous diffusion

coarse, weaker structures

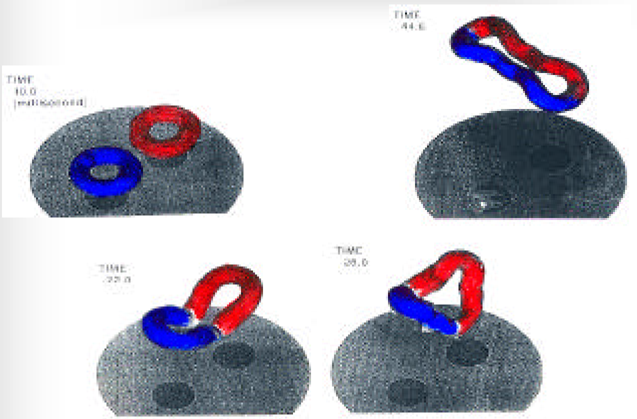
usually > 0, production (stretching)

⇒ finer, stronger structures

always < 0,

viscous dissipation

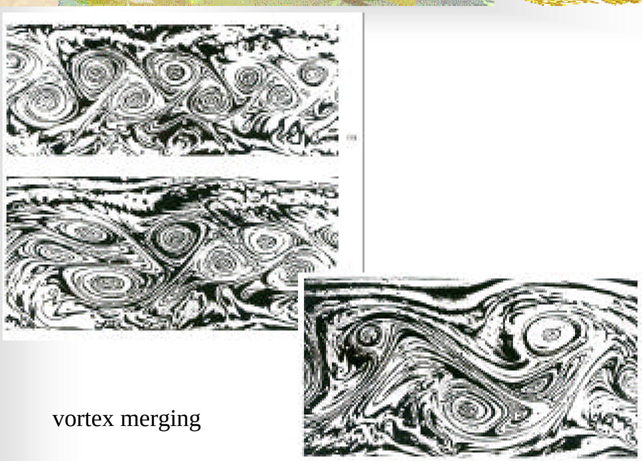
3. Vorticity and Vortex Stretching



3. Vorticity and Vortex Stretching

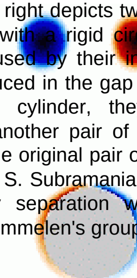
Ref: Couder and Basdevant, JFM 173, 225-251 (1986)

3. Vorticity and Vortex Stretching



vortex merging

3. Vorticity and Vortex Stretching



The image at the right depicts two counter-rotating vortices as they interact with a rigid circular cylinder. Their initial translation is caused by their interaction and the upward directed jet produced in the gap between them. Once they collide with the cylinder, they produce an unsteady separation and another pair of vortices having a rotation which opposes the original pair of vortices. This image was generated by Dr. S. Subramaniam and is undoubtedly part of the unsteady separation work being conducted by Professor van Dommelen's group in FSU.

3. Vorticity and Vortex Stretching

barotropic fluids, conservative body force:

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{u} + \nu\nabla^2\vec{\omega}$$

$$\frac{D}{Dt}\left(\tfrac{1}{2}\omega_i\omega_i\right) = \omega_i\omega_j\frac{\partial u_i}{\partial x_j} + \nu\frac{\partial}{\partial x_j}\left(\omega_i\frac{\partial \omega_i}{\partial x_j}\right) - \nu\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j}$$

- When a balance between the stretching and the diffusion/dissipation processes, the finest scale (Kolmogorov's, η) is reached.
- When ν is very small (or Reynolds number is extremely large), diffusion won't be significant until the gradients have become sufficiently large. That is, η decreases with decreasing ν.
- Will $\eta \to 0$ as $\nu \to 0$?

3. Vorticity and Vortex Stretching

Vortex stretching does not exist in two-dimensional flows.

$$\vec{u} = (u(x,y,t), v(x,y,t), 0)$$

$$\vec{\omega} = \nabla\times\vec{u} = (0,0,\omega(x,y,t))$$

$$(\vec{\omega}\cdot\nabla)\vec{u} = \omega\frac{\partial \vec{u}}{\partial z} = 0$$

⇨ Turbulence is always three-dimensional

3. Vorticity and Vortex Stretching

example

Townsend, Proc. Roy. Soc. A208, 534 (1951)

$$\vec{u} = \vec{U} + \vec{v} = (\alpha x, \beta y, -(\alpha+\beta)z) + \vec{v}$$

$$\nabla\cdot\vec{U} = 0 \ \text{(incompressible) and } \nabla\times\vec{U} = 0 \ \text{(irrotational)}$$

$$\vec{\omega} = \nabla\times\vec{u} = \nabla\times\vec{v}$$

Consider only special cases when $|\vec{v}| << |\vec{U}|$:

$$\text{convection}: \ (\vec{u}\cdot\nabla)\vec{\omega} \approx (\vec{U}\cdot\nabla)\vec{\omega}$$

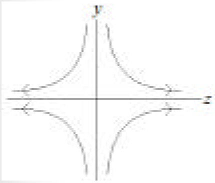
$$\text{stretching}: \ (\vec{\omega}\cdot\nabla)\vec{u} \approx (\vec{\omega}\cdot\nabla)\vec{U}$$

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{u} + \nu\nabla^2\vec{\omega}$$

$$\frac{\partial\vec{\omega}}{\partial t} + (\vec{U}\cdot\nabla)\vec{\omega} = (\vec{\omega}\cdot\nabla)\vec{U} + \nu\nabla^2\vec{\omega} \quad \text{(linear)}$$

3. Vorticity and Vortex Stretching

model (1): $\vec{U} = (0, -\alpha y, \alpha z)$



Vortex stretching:

$$(\vec{\omega}\cdot\nabla)\vec{U} = (0, -\alpha\omega_2, \alpha\omega_3)$$

Vorticity will be compressed in y- and stretched in z-direction.

Expect when steady, $\omega_3 = \omega_3(y) >> \omega_1, \omega_2 \approx 0$

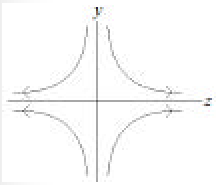
$$\frac{\partial\vec{\omega}}{\partial t} + (\vec{U}\cdot\nabla)\vec{\omega} = (\vec{\omega}\cdot\nabla)\vec{U} + \nu\nabla^2\vec{\omega}$$

$$\frac{\partial\omega_3}{\partial t} + (\vec{U}\cdot\nabla)\omega_3 = \omega_3\frac{\partial U_3}{\partial z} + \nu\nabla^2\omega_3 \quad \Rightarrow \quad -\alpha y\frac{d\omega_3}{dy} = \alpha\omega_3 + \nu\frac{d^2\omega_3}{dy^2}$$

3. Vorticity and Vortex Stretching

model (1): $\vec{U} = (0, -\alpha y, \alpha z)$

$$-\alpha y \frac{d\omega_3}{dy} = \alpha\omega_3 + \nu \frac{d^2\omega_3}{dy^2}$$



$$\Rightarrow \quad \omega_3(y) = \omega(y=0)\cdot \exp\left(-y^2 \Big/ \left(\tfrac{2\nu}{\alpha}\right)\right)$$

i.e. vortex sheet structures of thickness

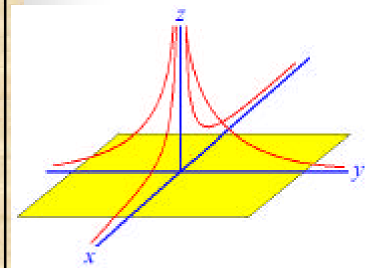
$$\sqrt{\frac{2\nu}{\alpha}} \sim \sqrt{\frac{\text{diffusion rate}}{\text{stretching rate}}}$$

Check:

$$\vec{\omega} = (0, 0, \omega_3(y)) \Rightarrow \vec{v} = (v_1(y), 0, 0) \quad \Rightarrow (\vec{\omega}\cdot\nabla)\vec{v} = (\vec{v}\cdot\nabla)\vec{\omega} = 0$$

3. Vorticity and Vortex Stretching

model (2): $\vec{U} = (-\alpha x, -\alpha y, 2\alpha z)$



Vortex stretching:

$$(\vec{\omega}\cdot\nabla)\vec{U} = (-\alpha\omega_1, -\alpha\omega_2, 2\alpha\omega_3)$$

Expect when steady,

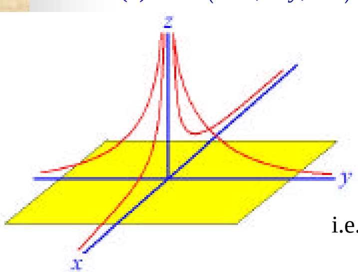
$$\omega_3 = \omega_3(r) >> \omega_1, \omega_2 \approx 0$$

$$\frac{\partial\omega_3}{\partial t} + (\vec{U}\cdot\nabla)\omega_3 = \omega_3\frac{\partial U_3}{\partial z} + \nu\nabla^2\omega_3$$

$$\Rightarrow \; -\alpha r\frac{d\omega_3}{dr} = 2\alpha\omega_3 + \frac{\nu}{r}\frac{d}{dr}\left(r\frac{d\omega_3}{dr}\right)$$

3. Vorticity and Vortex Stretching

model (2): $\vec{U} = (-\alpha x, -\alpha y, 2\alpha z)$



$$-\alpha r\frac{d\omega_3}{dr} = 2\alpha\omega_3 + \frac{\nu}{r}\frac{d}{dr}\left(r\frac{d\omega_3}{dr}\right)$$

$$\Rightarrow \quad \omega_3(r) = \omega_3(r=0)\cdot \exp\left(-r^2 \Big/ \tfrac{2\nu}{\alpha}\right)$$

i.e. vortex tube structures of radius

$$\sqrt{\frac{2\nu}{\alpha}} \sim \sqrt{\frac{\text{diffusion rate}}{\text{stretching rate}}}$$

Check:

$$\vec{\omega} = (0, 0, \omega_3(r)) \Rightarrow \vec{v} = (0, v_\theta(r), 0) \quad \Rightarrow (\vec{\omega}\cdot\nabla)\vec{v} = (\vec{v}\cdot\nabla)\vec{\omega} = 0$$

3. Vorticity and Vortex Stretching

example

Buntine and Pullin, JFM 205, 263 (1989)

$$\vec{u} = \vec{U} + \vec{v} = (\alpha x, \beta y, -(\alpha+\beta)z) + \vec{v}$$

$\nabla\cdot\vec{U} = 0$ (incompressible) and $\nabla\times\vec{U} = 0$ (irrotational)

$$\vec{\omega} = \nabla\times\vec{u} = \nabla\times\vec{v}$$

Consider only special cases when $\vec{v}$ is two-dimensional, i.e..

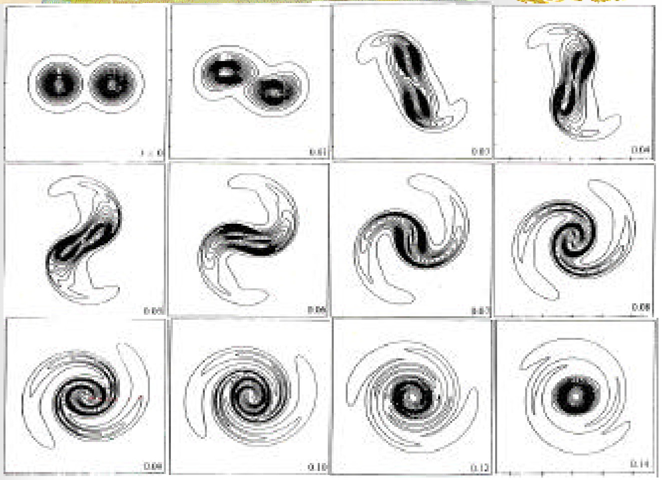
$$\vec{v} = \left(\; v_1(x,y,t), v_2(x,y,t), 0 \;\right)$$

$$\vec{\omega} = \left(\; 0, 0, \omega(x,y,t) \;\right)$$

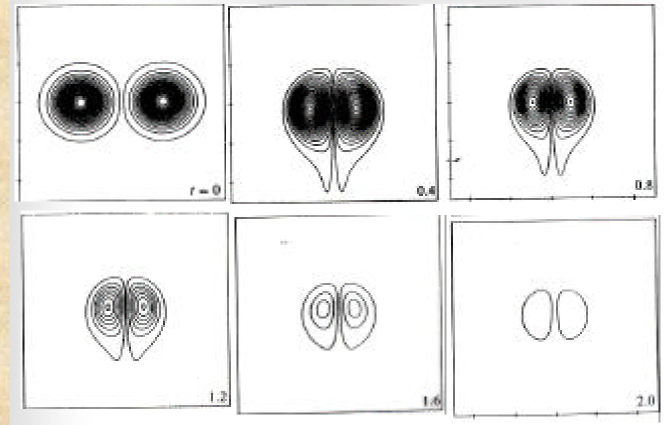
vortex stretching: $(\vec{\omega}\cdot\nabla)\vec{u} = (\vec{\omega}\cdot\nabla)(\vec{U}+\vec{v}) = (\vec{\omega}\cdot\nabla)\vec{U}$

convection: $(\vec{u}\cdot\nabla)\vec{\omega} = \{(\vec{U}+\vec{v})\cdot\nabla\}\vec{\omega} \neq (\vec{U}\cdot\nabla)\vec{\omega}$

3. Vorticity and Vortex Stretching



3. Vorticity and Vortex Stretching



3. Vorticity and Vortex Stretching

example

T.S. Lundgren, Phys. Fluids 25, 2193 (1982)

$$\vec{U} = \left(U_r = -a(t)r/2,\ U_\theta = 0,\ U_z = a(t)z \right)$$

~ same with those used by Buntine and Pullin

~ analytical solutions

~ tube core surrounded by spiral vortex sheets

Nonlinearity Generates Scales

$$\nabla \cdot \vec{u} = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla p + \vec{g} + \frac{\mu}{\rho}\nabla^2 \vec{u}$$

If at some instant time, the flow has

$$u = A\cos kx$$

$$v = B\cos mx$$

then,

$$u\frac{\partial u}{\partial x} = -A\cos kx \cdot Ak\sin kx \quad = -\frac{A^2 k}{2}\sin(2kx)$$

$$u\frac{\partial v}{\partial x} = -A\cos kx \cdot Bm\sin mx \quad = -\frac{ABm}{2}\left(\sin(m+k)x + \sin(m-k)x\right)$$