7. Classical Models of Jets, Wakes, and Boundary Layers

thin boundary layer approximation:

• laminar boundary layers: $\mathrm{Re} = \frac{UL}{\nu} = \frac{L^2/\nu}{L/U} >> 1$

• turbulent boundary layers: $\frac{O(u')}{O(\bar{u})} << 1$

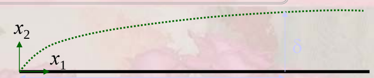
Turbulent velocities are found to be on the order of $u_* \equiv \sqrt{\tau_w/\rho}$ in both the viscous sublayer and outer layer.

$$0 = \frac{\partial \bar{u}_j}{\partial x_j} = \frac{\partial \bar{u}_1}{\partial x_1} + \frac{\partial \bar{u}_2}{\partial x_2} + \underbrace{\cancel{\frac{\partial \bar{u}_3}{\partial x_3}}}_{\text{2D}}$$

incompressible

$$\underbrace{\cancel{\frac{\partial \bar{u}_i}{\partial t}}}_{\text{steady}} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \cancel{\frac{\partial \bar{u}_j}{\partial x_i}}\right) - \rho\overline{u_i' u_j'}\right)$$

7. Classical Models of Jets, Wakes, and Boundary Layers



$$0 = \frac{\partial \bar{u}_1}{\partial x_1} + \frac{\partial \bar{u}_2}{\partial x_2} \Rightarrow \frac{U_1}{d} \sim \frac{O(\bar{u}_2)}{\delta} \quad \text{or} \quad O(\bar{u}_2) \sim U_1\frac{\delta}{d}$$

$$\bar{u}_j \frac{\partial \bar{u}_1}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_1} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu\frac{\partial \bar{u}_1}{\partial x_j} - \rho\overline{u_1' u_j'}\right)$$

$\because q << U_1$

$$\bar{p} \sim \rho U_1^2 \Leftarrow \frac{U_1^2}{d} \sim -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_1} + \cancel{\frac{\nu U_1}{\delta^2}} - \cancel{\frac{q^2}{d}} - \frac{q^2}{\delta}$$

assume large Reynolds number
reasonable except near wall

$$\frac{U_1^2}{d} \sim \frac{q^2}{\delta} \Rightarrow \frac{\delta}{d} \sim \frac{q^2}{U_1^2} \quad \left(\text{in fact} \le \frac{q^2}{U_1^2}\right)$$

$$O(\bar{u}_2) << q \sim U_1\sqrt{\frac{\delta}{d}} << U_1$$

$$\bar{u}_1 \frac{\partial \bar{u}_1}{\partial x_1} + \bar{u}_2 \frac{\partial \bar{u}_1}{\partial x_2} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_1} - \frac{\partial \overline{u_1' u_2'}}{\partial x_2}$$

7. Classical Models of Jets, Wakes, and Boundary Layers

$$\bar{u}_j \cancel{\frac{\partial \bar{u}_2}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_2} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu\cancel{\frac{\partial \bar{u}_2}{\partial x_j}} - \rho\overline{u_2' u_j'}\right)$$

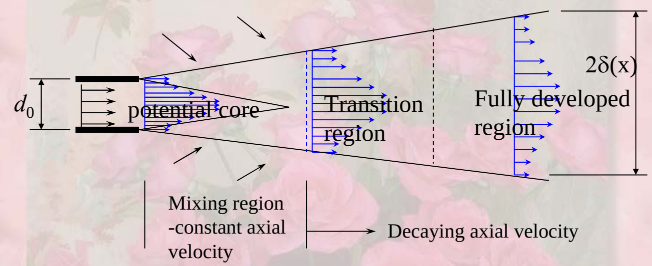
$$\bar{u}_j \frac{\partial \bar{u}_2}{\partial x_j} \sim \frac{U_1^2}{d}\frac{\delta}{d} \quad << \quad \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_2} \sim \frac{U_1^2}{\delta} \qquad \boxed{\bar{p} \sim \rho U_1^2}$$

$$\frac{\partial \overline{u_2' u_j'}}{\partial x_j} = \frac{\partial \overline{u_1' u_2'}}{\partial x_1} + \frac{\partial \overline{u_2'^2}}{\partial x_2} \sim \frac{q^2}{d} + \frac{q^2}{\delta} \sim \frac{U_1^2}{d} \qquad \boxed{q \sim U_1\sqrt{\frac{\delta}{d}}}$$

$\Rightarrow \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_2} \approx 0$ in the sense that it is compared with the leading terms in problem

$\Rightarrow \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_2} - \frac{\partial \overline{u_2'^2}}{\partial x_2} \approx 0$ ~ second-order terms

7. Plane Jet



Fully Developed Region (at distances of about $20d_0$ and greater)

– Velocity profile shape becomes more or less “unchanged” (similar or self-preserving)

7. Plane Jet

$$\bar{u}_1\left(\frac{\partial \bar{u}_1}{\partial x_1}+\frac{\partial \bar{u}_2}{\partial x_2}=0\right)$$

$$+)\quad \bar{u}_1\frac{\partial \bar{u}_1}{\partial x_1}+\bar{u}_2\frac{\partial \bar{u}_1}{\partial x_2}=-\frac{1}{\rho}\cancel{\frac{dp}{dx_1}}-\frac{\partial \overline{u_1'u_2'}}{\partial x_2}$$

$$\int_{-\infty}^{\infty}\frac{\partial \bar{u}_1^2}{\partial x_1}+\frac{\partial \bar{u}_1\bar{u}_2}{\partial x_2}=-\frac{\partial \overline{u_1'u_2'}}{\partial x_2}\,dx_2$$

$$\Rightarrow\int_{-\infty}^{\infty}\frac{\partial \bar{u}_1^2}{\partial x_1}dx_2=\frac{\partial}{\partial x_1}\int_{-\infty}^{\infty}\bar{u}_1^2dx_2=0$$

- constant momentum flux:

$$J\equiv\rho\int_{-\infty}^{\infty}\bar{u}_1^2dy=\text{momentum flux}=\text{constant}\quad(\text{kg/sec}^2)$$

- Dimensional analysis:

$$U_1=U_1(x_1,J/\rho)\qquad \frac{U_1}{\sqrt{J/\rho x_1}}=\text{constant}\quad\Rightarrow\quad U_1\propto x_1^{-1/2}$$

$$\delta=\delta(x_1,J/\rho)\qquad \frac{\delta}{x_1}=\text{constant}\quad\Rightarrow\quad \delta\propto x_1$$

7. Classical Models of Jets, Wakes, and Boundary Layers

one-point model --- turbulent eddy viscosity ν_t

$$\frac{\tau_{ij}^t}{\rho}=-\overline{u_i'u_j'}=\nu_t\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)-\frac{2}{3}\delta_{ij}K$$

~ consistent with $-\overline{u_i'u_i'}=\nu_t\left(\frac{\partial \bar{u}_i}{\partial x_i}+\frac{\partial \bar{u}_i}{\partial x_i}\right)-\frac{2}{3}\delta_{ii}K=0-2K$

~ by analogy with the kinetic theory of gases, but the clear separation of scales between microscopic and macroscopic which underlines kinetic theory is not present in turbulence

~ isotropic (turbulent velocity components have all equal mean-square values)

$$\frac{\tau_{12}^t}{\rho}=-\overline{u_1'u_2'}=\nu_t\left(\frac{\partial \bar{u}_1}{\partial x_2}+\frac{\partial \bar{u}_2}{\partial x_1}\right)\approx\nu_t\frac{\partial \bar{u}_1}{\partial x_2}$$

7. Plane Jet

$$\frac{\partial \bar{u}_1}{\partial x_1}+\frac{\partial \bar{u}_2}{\partial x_2}=0$$

$$\bar{u}_1\frac{\partial \bar{u}_1}{\partial x_1}+\bar{u}_2\frac{\partial \bar{u}_1}{\partial x_2}=-\frac{\partial}{\partial x_2}\left(\nu_t\frac{\partial \bar{u}_1}{\partial x_2}\right)$$

- similarity solution:

$$\bar{u}_1(x_1,x_2)=U_1(x_1)F'(x_2/\delta)=U_1(x_1)F'(\eta)$$

$$\bar{\psi}(x_1,x_2)=U_1(x_1)\delta(x_1)F(\eta)$$

$$\bar{u}_2(x_1,x_2)=-F(\eta)\frac{d}{dx_1}(U_1\delta)+(U_1\delta')(F'\eta)$$

fn. Of η only

$$(U_1U_1')F'^2-(U_1\delta)'\frac{U_1}{\delta}FF''=\nu_t\frac{U_1}{\delta^2}F'''$$

fn. of x_1 only

Assume: $\nu_t=\nu_t(x_1)$

7. Plane Jet

$$\bar{u}_1=U_1(x_1)F'(\eta)$$

$$J\equiv\rho\int_{-\infty}^{\infty}\bar{u}_1^2dy=\text{constant}=\rho U_1^2\delta\int_{-\infty}^{\infty}F'^2d\eta$$

$$\frac{(U_1\delta)'\frac{U_1}{\delta}}{U_1U_1'}=1+\frac{U_1\delta'}{U_1'\delta}=\text{constant}\equiv a\;=-1$$

$$\frac{\nu_t\frac{U_1}{\delta^2}}{U_1U_1'}=\frac{\nu_t}{\delta^2U_1'}=\text{constant}\equiv b\Rightarrow\nu_t=b\delta^2U_1'\propto x_1^{1/2}$$

$$U_1^2\delta=\text{constant}\quad\left(U_1\propto x_1^{-1/2},\ \delta\propto x_1\right)$$

$$\bar{u}_1(x_2=0)=U_1(x_1)$$
$$\bar{u}_1(x_2\to\infty)=0$$
$$\frac{\partial \bar{u}_1}{\partial x_2}(x_2=0)=0$$

choose $\delta=4Bx_1$ and $b=-\frac{1}{2}\Rightarrow\nu_t=B\delta U_1$

$$F'^2+FF''+\frac{1}{2}F'''=0$$
$$F'(0)=1$$
$$F'(\infty)=0$$
$$F''(0)=0$$

7. Plane Jet

$F(\eta) = \tanh \eta$

$$\bar{u}_1 = U_1(x_1)\operatorname{sech}^2\eta = A x_1^{-1/2}\operatorname{sech}^2\left(\frac{x_2}{4Bx_1}\right)$$

$$\bar{u}_2(x_1, \pm\infty) = \mp 2BAx_1^{-1/2}$$

(entrainment)

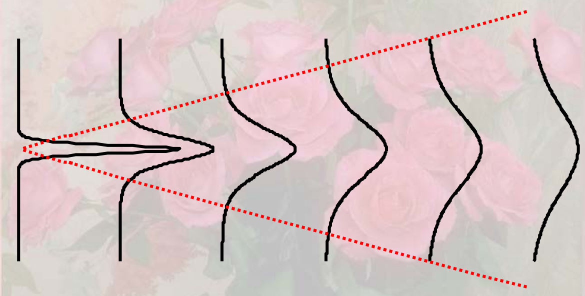
A and B are determined by the jet momentum flux J and the rate of spreading.

$$J = \rho U_1^2 \delta \int_{-\infty}^{\infty} F'^2 d\eta = \frac{32}{3}\rho A^2 B$$

$$\delta = 4Bx_1$$

7. Plane Jet

$$\bar{u}_1 = U_1(x_1)\operatorname{sech}^2\eta = A x_1^{-1/2}\operatorname{sech}^2\left(\frac{x_2}{4Bx_1}\right)$$



7. Circular Jet

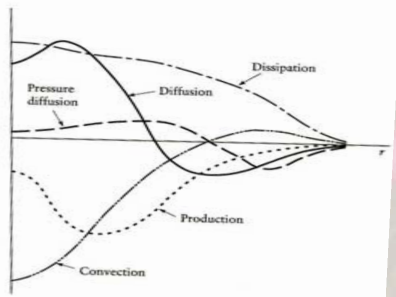
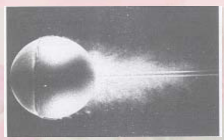
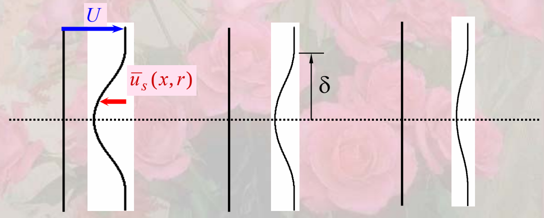


Figure 5.8. Different terms in the turbulent energy balance of circular jet in the self-similar regime. Note that the sign of the pro duction term has been switched here to make the contributior shown sum to zero. The viscous contribution to the diffusive ter in (5.41) is negligible outside the viscous sublayer, as here, while th remainder is the sum of a pressure–velocity term ("Pressure diff sion" in the figure) and a cubic velocity term ("Diffusion" in th figure). (Wygnanski and Fiedler (1969), redrawn.)

7. Round Wake



Assumptions: steady, axisymmetric, $\nu_t >> \nu$



$\bar{u}_s(x,r) << U$

$\bar{u}_1 = U - \bar{u}_s(x,r)$

7. Round Wake

$$\int_0^\infty 2\pi r\left\{\frac{\partial \bar u_1}{\partial x}+\frac{1}{r}\frac{\partial r\bar u_2}{\partial r}=0\right\}dr$$

constant volume flux: $\frac{d}{dx}\int_0^\infty 2\pi r\bar u_1\,dr=0$

$\bar u_1 = U-\bar u_s(x,r)\Rightarrow$ invariant : $Q=\int_0^\infty 2\pi r\bar u_s\,dr$ = constant

$$-\frac{\partial \bar u_s}{\partial x}+\frac{1}{r}\frac{\partial r\bar u_2}{\partial r}=0$$

$$\frac{O(\bar u_s)}{x}\sim\frac{O(\bar u_2)}{\delta}\Rightarrow O(\bar u_2)\sim\frac{\delta}{x}O(\bar u_s)$$

7. Round Wake

$$\bar u_1\frac{\partial\bar u_1}{\partial x}+\bar u_2\frac{\partial \bar u_1}{\partial r}=\frac{1}{r}\frac{\partial}{\partial r}\left(\nu_t r\frac{\partial\bar u_1}{\partial r}\right)$$

$$(U-\cancel{\bar u_s})\frac{\partial \bar u_s}{\partial x}+\cancel{\bar u_2\frac{\partial\bar u_s}{\partial r}}=\frac{1}{r}\frac{\partial}{\partial r}\left(\nu_t r\frac{\partial \bar u_s}{\partial r}\right)$$

$\bar u_s << U$

$$\frac{UO(\bar u_s)}{x}+\frac{\delta O(\bar u_s)}{x}\frac{O(\bar u_s)}{\delta}\sim\frac{\nu_t O(\bar u_s)}{\delta^2}$$

$$U\frac{\partial\bar u_s}{\partial x}=\frac{1}{r}\frac{\partial}{\partial r}\left(\nu_t r\frac{\partial\bar u_s}{\partial r}\right)$$

- similarity solution: $\bar u_s=\bar u_0(x)F(r/\delta(x))=\bar u_0(x)F(\eta)$

$Q=\int_0^\infty 2\pi r\bar u_s\,dr$ = constant $\Rightarrow$ $\bar u_0\delta^2$ = constant

7. Round Wake

$$U\frac{\partial\bar u_s}{\partial x}=\frac{1}{r}\frac{\partial}{\partial r}\left(\nu_t r\frac{\partial\bar u_s}{\partial r}\right)$$

$\bar u_s=\bar u_0(x)F(\eta)$ ⇩ ν_t = fn. of x only

$$U\left(\bar u_0' F-\frac{\bar u_0\delta'}{\delta}\eta F'\right)=\frac{\nu_t\bar u_0}{\delta^2}\left(\frac{F'}{\eta}+F''\right)$$

⇩

$\frac{\bar u_0\delta'}{\bar u_0'\delta}$ = constant $=-\frac{1}{2}$

$\frac{\nu_t}{U\delta^2}\frac{\bar u_0}{\bar u_0'}$ = constant $=-1$ (so chosen)

$\bar u_0\delta^2$ = constant

⇨ $\left(F+\frac{1}{2}\eta F'\right)=-\left(\frac{F'}{\eta}+F''\right)$ with $F(0)=1, F(\infty)=0$

7. Round Wake

$$F(\eta)=\exp\left(-\tfrac{1}{2}\eta^2\right)$$

- turbulent stress $-\overline{u_1'u_2'}=\nu_t\frac{\partial\bar u_1}{\partial r}=-\nu_t\frac{\partial\bar u_s}{\partial r}$

expect $\bar u_0^2\sim\nu_t\frac{\bar u_0}{\delta}$ $\Rightarrow$ $\nu_t\sim\delta\bar u_0$

recall $\frac{\nu_t}{U\delta^2}\frac{\bar u_0}{\bar u_0'}=-1$

$\bar u_0\delta^2$ = constant

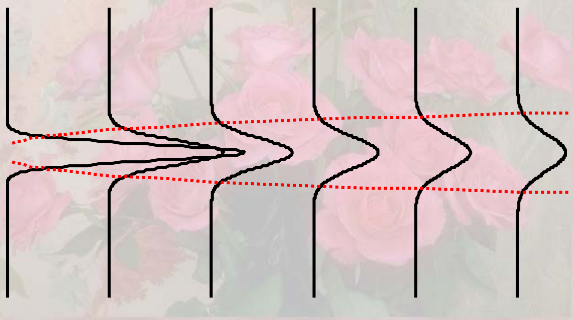
$$\Rightarrow\left\{\begin{array}{l}\bar u_0\propto x^{-2/3}\\ \delta\propto x^{1/3}\\ \nu_t\propto x^{-1/3}\end{array}\right.$$

$$\bar u_s=\bar u_0(x)F(\eta)=Ax^{-2/3}\exp\left(-\frac{1}{2}\left(\frac{r}{Bx^{1/3}}\right)^2\right)$$

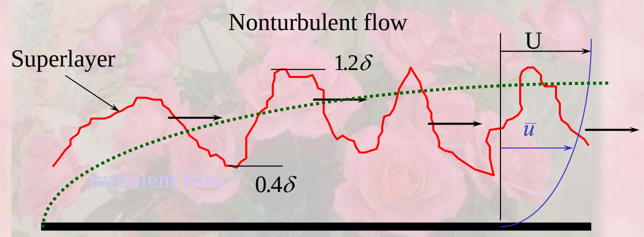
$$Q=\int_0^\infty 2\pi r\bar u_s\,dr=2\pi\bar u_0\delta^2=2\pi AB^2$$

7. Round Wake

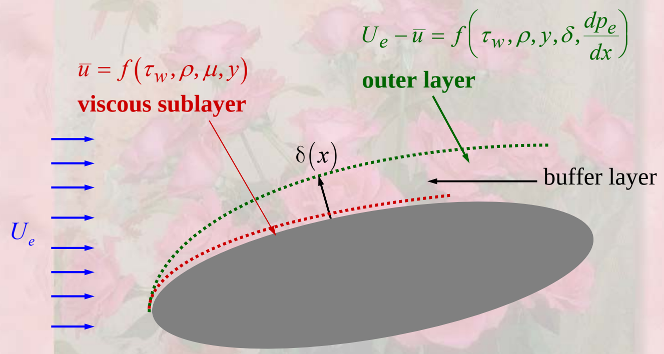
$$\bar{u}_s = \bar{u}_0(x)F(\eta) = Ax^{-2/3}\exp\left(-\frac{1}{2}\left(\frac{r}{Bx^{1/3}}\right)^2\right)$$



7. Boundary Layer



7. Boundary Layer



7. Boundary Layer

Dimensionless Velocity Profiles

- Inner law: $\bar{u} = f(\tau_w, \rho, \mu, y)$

$$\frac{\bar{u}}{u_*} = f\left(\frac{yu_*}{\nu}\right) \qquad u_* = \left(\frac{\tau_w}{\rho}\right)^{1/2}$$

wall-friction velocity
(shear velocity)

- Outer law: $U_e - \bar{u} = f\left(\tau_w, \rho, y, \delta, \frac{dp_e}{dx}\right)$

$$\frac{U_e - \bar{u}}{u_*} = g\left(\frac{y}{\delta}, \xi\right) \qquad \xi = \frac{\delta}{\tau_w}\frac{dp_e}{dx}$$

velocity-defect law

- Overlap law:

$$\frac{\bar{u}}{u_*} = f\left(\frac{\delta\, u_*}{\nu}\frac{y}{\delta}\right) = \frac{U_e}{u_*} - g\left(\frac{y}{\delta}\right)$$

7. Boundary Layer

Logarithmic Velocity Profiles (in the buffer layer)

$$\frac{\bar{u}}{u_*} = f\left(\frac{\delta\, u_*}{\nu}\frac{y}{\delta}\right) = \frac{U_e}{u_*} - g\left(\frac{y}{\delta}\right)$$

$$f\left(\frac{\delta\, u_*}{\nu}\frac{y}{\delta}\right) = g_1\left(\frac{\delta\, u_*}{\nu}\right) - g_2\left(\frac{y}{\delta}\right)$$

logarithmic functions!

7. Boundary Layer

Logarithmic Velocity Profiles

Inner variables : $\frac{\bar{u}}{u_*} = f\left(\frac{y u_*}{\nu}\right) = \frac{1}{\kappa}\ln\frac{y u_*}{\nu} + B$

Outer variables : $\frac{U_e - \bar{u}}{u_*} = g\left(\frac{y}{\delta}\right) = -\frac{1}{\kappa}\ln\frac{y}{\delta} + A$

$\kappa \approx 0.41$ $B \approx 5.0$

Flat plate, **A = 2.5**

Strong favorable, **A = 1.0**

Mild adverse, **A = 5.6**

Strong adverse, **A = 13**

7. Boundary Layer

$\frac{\bar{u}_x}{u_*}$

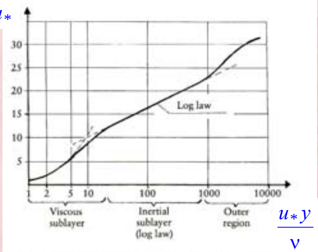


Figure 5.12. A typical velocity distribution across the whole of a turbulent boundary layer. The vertical axis is linear, the horizontal one is logarithmic. Note the wide range of values of y, which increases with Reynolds number.

$\frac{u_* y}{\nu} = y^+$

$u_* \equiv \sqrt{\frac{\tau_w}{\rho}}$

$\frac{u_*}{\bar{u}_x}$ small ~ a consequence of high Reynolds number

$$\frac{\bar{u}_x}{u_*} = \begin{cases} y^+ & \text{in viscous sublayer} \\ \frac{1}{\kappa}\ln y^+ + a & \text{in inertial sublayer} \\ \frac{1}{\kappa}\ln\frac{y}{\delta} + b\left(1 - \frac{1}{2}\varpi\left(\frac{y}{\delta}\right)\right) & \text{in outer region} \end{cases}$$

7. Boundary Layer

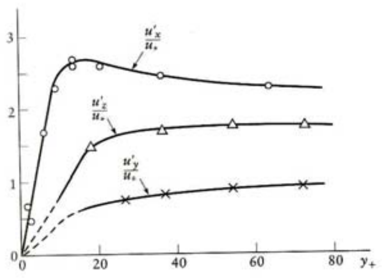


Figure 5.14. Root-mean-squared turbulent velocities in a viscous sublayer, scaled on u_* (Laufer (1954), redrawn.)

7. Boundary Layer

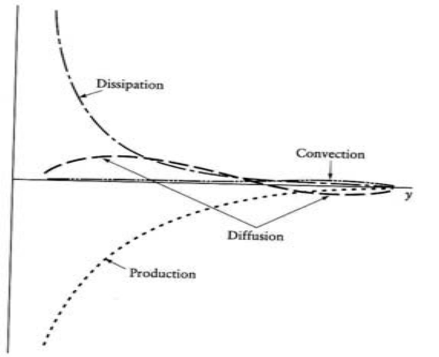


Figure 5.15. Terms in the turbulent energy balance for a flat-plate boundary layer at zero incidence. The sign of the production term has been switched so the different contributions sum to zero. The curve labeled diffusion incorporates both the pressure and cubic velocity diffusive terms of (5.43). (Klebanoff (1955), redrawn.)

8. Spectral Analysis of Homogeneous Turbulence

homogeneous $\Rightarrow$ $\langle \vec{u} \rangle$ = constant

assume $\bar{u}_i = 0$ for convenience $\Rightarrow$ $u_i = u'_i$

$$u(\vec{x},t) = \int \hat{u}(\vec{k},t)\exp(i\vec{k}\cdot\vec{x})d\vec{k}$$

$\hat{u}(\vec{k},t)d\vec{k}$ = amplitude of mode of wave vector $\vec{k}$

$\frac{2\pi}{|\vec{k}|} = \frac{2\pi}{k}$ = wave length ~ ℓ = spatial scale represented by $\vec{k}$

Turbulent spatial scales of $O(\ell)$ will mostly contribute to the spectrum for k of $O(\ell^{-1})$, but will have effects at all k.

$k_{\min} \sim L^{-1}$ and $k_{\max} \sim \eta^{-1}$

8. Spectral Analysis of Homogeneous Turbulence

two-point velocity correlations

$$R_{ij}(\vec{x},\vec{x}',t) = \overline{u_i(\vec{x},t)u_j(\vec{x}',t)} = R_{ij}(\vec{x}-\vec{x}',t) = R_{ij}(\vec{r},t) \equiv \int \Phi_{ij}(\vec{k},t)e^{i\vec{k}\cdot\vec{r}}d\vec{k}$$

$\triangleright$ $\Phi_{ij}(\vec{k},t)$ is a Hermitian matrix: $\Phi_{ji}(\vec{k},t) = \Phi^*_{ij}(\vec{k},t)$

$$R_{ij}(\vec{x}-\vec{x}',t) = \overline{u_i(\vec{x},t)u_j(\vec{x}',t)} = \overline{u_j(\vec{x}',t)u_i(\vec{x},t)} = R_{ji}(-\vec{r},t)$$

$$\Phi_{ji}(\vec{k},t) = \frac{1}{(2\pi)^3}\int R_{ji}(\vec{r},t)e^{-i\vec{k}\cdot\vec{r}}d\vec{r} = \frac{1}{(2\pi)^3}\int R_{ji}(-\vec{r},t)e^{i\vec{k}\cdot\vec{r}}d\vec{r}$$

$$= \frac{1}{(2\pi)^3}\int \underbrace{R_{ij}}_{\text{real}}(\vec{r},t)e^{i\vec{k}\cdot\vec{r}}d\vec{r} = \Phi^*_{ij}(\vec{k},t)$$

$\Rightarrow$ $\Phi_{11}(\vec{k},t), \Phi_{22}(\vec{k},t), \Phi_{33}(\vec{k},t)$ all real

8. Spectral Analysis of Homogeneous Turbulence

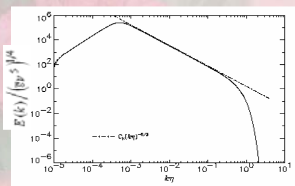
$\triangleright$ $\frac{1}{2}\Phi_{ii}(\vec{k},t)$ = (real) energy density of wave vector $\vec{k}$

$$R_{ij}(\vec{x},\vec{x}',t) = \overline{u_i(\vec{x},t)u_j(\vec{x}',t)} \equiv \int \Phi_{ij}(\vec{k},t)e^{i\vec{k}\cdot\vec{r}}d\vec{k}$$

$$\frac{1}{2}\overline{u_i u_i} = \frac{1}{2}R_{ii}(0,t) = \int \frac{1}{2}\Phi_{ii}(\vec{k},t)d\vec{k}$$

$$= \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{1}{2}\Phi_{ii}(\vec{k},t)k^2 \sin\theta d\theta d\phi\, dk$$

$$= \int_0^\infty E(k)dk$$



$$E(k) = \int_0^{2\pi} \int_0^\pi \frac{1}{2}\Phi_{ii}(\vec{k},t)k^2 \sin\theta d\theta d\phi$$

isotropic: $E(k) = 2\pi k^2 \Phi_{ii}(\vec{k},t)$

8. Spectral Analysis of Homogeneous Turbulence

velocity $u(\vec{x},t)=\int \hat{u}(\vec{k},t)\exp(i\vec{k}\cdot\vec{x})d\vec{k}$

$$\hat{u}_i(\vec{k},t)=\frac{1}{(2\pi)^3}\int_{X^3} u_i(\vec{x},t)\exp(-i\vec{k}\cdot\vec{x})d\vec{x}$$

~ do not converge

~ random $\hat{u}_i(\vec{k},t)$ with zero mean and variance ~ $X^{3/2}$

$$\sim\ \Phi_{ij}(\vec{k},t)=\lim_{X\to\infty}\left\{\left(\frac{\pi}{X}\right)^3\overline{\hat{u}_i(\vec{k},t)\hat{u}_j(\vec{k},t)}\right\}$$

$$\sim\ ik_j\hat{u}_i=\frac{1}{(2\pi)^3}\int\frac{\partial u_i}{\partial x_j}e^{-i\vec{k}\cdot\vec{x}}d\vec{x}$$

$$\sim\ \hat{u}_i\otimes\hat{u}_j\equiv\int\hat{u}_i(\vec{p},t)\hat{u}_j\left(\vec{k}-\vec{p},t\right)d\vec{p}=\frac{1}{(2\pi)^3}\int u_iu_je^{-i\vec{k}\cdot\vec{x}}d\vec{x}$$

8. Spectral Analysis of Homogeneous Turbulence

▷ $\overline{\hat{u}_i(\vec{k},t)\hat{u}_j(\vec{k}',t)}=\Phi_{ij}(\vec{k},t)\delta\left(\vec{k}+\vec{k}'\right)$

$$\overline{\hat{u}_i(\vec{k},t)\hat{u}_j(\vec{k}',t)}=\frac{1}{(2\pi)^6}\int\int\overline{u_i(\vec{x},t)u_j(\vec{x}',t)}e^{-i\vec{k}\cdot\vec{x}}e^{-i\vec{k}'\cdot\vec{x}'}d\vec{x}d\vec{x}'$$

$$=\frac{1}{(2\pi)^6}\int\int R_{ij}(\vec{r}=\vec{x}-\vec{x}',t)e^{-i\vec{k}\cdot(\vec{x}'+\vec{r})}e^{-i\vec{k}'\cdot\vec{x}'}d\vec{r}d\vec{x}'$$

$$=\frac{1}{(2\pi)^3}\int\Phi_{ij}(\vec{k},t)e^{-i\left(\vec{k}+\vec{k}'\right)\cdot\vec{x}'}d\vec{x}'$$

$$=\Phi_{ij}(\vec{k},t)\delta\left(\vec{k}+\vec{k}'\right)$$

For homogeneous turbulence, there is no correlation in Fourier space between two wave vectors whose sum is not zero.

8. Spectral Analysis of Homogeneous Turbulence

Vorticity: $\vec{\omega}=\nabla\times\vec{u}\ \Rightarrow\ \hat{\omega}_i=\varepsilon_{ijm}ik_j\hat{u}_m\ \Rightarrow\ \hat{\omega}_i\hat{\omega}_i^*=k^2\hat{u}_i\hat{u}_i^*$

$$\overline{u_i(\vec{x},t)u_j(\vec{x}',t)}\equiv R_{ij}(\vec{r},t)=\int\Phi_{ij}(\vec{k},t)e^{i\vec{k}\cdot\vec{r}}d\vec{r}$$

$$\overline{\hat{u}_i(\vec{k},t)\hat{u}_j(\vec{k}',t)}=\Phi_{ij}(\vec{k},t)\delta\left(\vec{k}+\vec{k}'\right)$$

$\frac{1}{2}\overline{u_iu_i}=\int\frac{1}{2}\Phi_{ii}(\vec{k},t)d\vec{k}=\int_0^\infty E(k,t)dk$ (energy spectrum)

$$\overline{\omega_i(\vec{x},t)\omega_j(\vec{x}',t)}\equiv R_{ij}^{\omega}(\vec{r},t)=\int\Phi_{ij}^{\omega}(\vec{k},t)e^{i\vec{k}\cdot\vec{r}}d\vec{r}$$

$$\overline{\hat{\omega}_i(\vec{k},t)\hat{\omega}_j(\vec{k}',t)}=\Phi_{ij}^{\omega}(\vec{k},t)\delta\left(\vec{k}+\vec{k}'\right)$$

$\frac{1}{2}\overline{\omega_i\omega_i}=\int\frac{1}{2}\Phi_{ii}^{\omega}(\vec{k},t)d\vec{k}=\int_0^\infty\Omega(k,t)dk$ **(enstrophy spectrum)**

$$\overline{\hat{\omega}_i\hat{\omega}_i^*}=k^2\overline{\hat{u}_i\hat{u}_i^*}\ \Rightarrow\ \Phi_{ii}^{\omega}(\vec{k},t)=k^2\Phi_{ii}(\vec{k},t)\Rightarrow\boxed{\Omega(k,t)=k^2E(k,t)}$$

8. Spectral Analysis of Homogeneous Turbulence

mean energy dissipation rate

$$\bar{\varepsilon}=2\nu\overline{S_{ij}S_{ij}}$$

$$=\nu\left(\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}+\cancel{\frac{\partial^2\overline{u_iu_j}}{\partial x_i\partial x_j}}\right)=\nu\left(\overline{\omega_i\omega_i}+2\cancel{\frac{\partial^2\overline{u_iu_j}}{\partial x_i\partial x_j}}\right)$$

enstrophy spectrum: $\frac{1}{2}\overline{\omega_i\omega_i}=\int_0^\infty\Omega(k,t)dk=\int_0^\infty k^2E(k)dk$

$$\bar{\varepsilon}=2\nu\int_0^\infty\Omega(k,t)dk=2\nu\int_0^\infty k^2E(k,t)dk$$

~ smaller eddies with higher weighting

velocity Fourier transform

$$ik_i\hat{u}_i(\vec{k},t) = \frac{1}{(2\pi)^3}\int_{X^3}\frac{\partial u_i}{\partial x_i}(\vec{x},t)\exp(-i\vec{k}\cdot\vec{x})d\vec{x} = 0$$

$\Downarrow$

$k_i\hat{u}_i(\vec{k},t) = 0$ The Fourier transform of the velocity is perpendicular to the wave vector.

$$\frac{\partial u_i}{\partial t} + \frac{\partial(u_iu_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\mu}{\rho}\frac{\partial^2 u_i}{\partial x_j\partial x_j}$$

no turbulent production because of the absence of the mean motion

$\Downarrow$

$$\boxed{\frac{\partial\hat{u}_i}{\partial t} + ik_j(\hat{u}_i\otimes\hat{u}_j) = -ik_i\frac{\hat{p}}{\rho} - \nu k^2\hat{u}_i} \qquad \boxed{\frac{\hat{p}}{\rho} = -\frac{k_ik_j}{k^2}(\hat{u}_i\otimes\hat{u}_j)}$$

$$ik_ik_j(\hat{u}_i\otimes\hat{u}_j) = -ik_ik_i\frac{\hat{p}}{\rho} = -ik^2\frac{\hat{p}}{\rho}$$

$$\frac{\partial\hat{u}_i}{\partial t} + i\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)k_m(\hat{u}_j\otimes\hat{u}_m) = -\nu k^2\hat{u}_i$$

$\Delta_{ij} \equiv \delta_{ij} - \frac{k_ik_j}{k^2}$ = a projection operator: project any vector onto the plane perpendicular to $\vec{k}$

$$\Delta_{ij}a_j \equiv \delta_{ij}a_j - \frac{k_ik_j}{k^2}a_j = a_i - \underbrace{\frac{k_i}{k}\frac{k_ja_j}{k}}_{\text{component of vector a in the direction of } \vec{k}}$$

~ the compressibility of the velocity field induced by the nonlinear term is balanced by the pressure term

~ the spectral nonlocality arises from the convolution term

governing equation for the spectral correlation $\Phi_{ij}(\vec{k},t)$

$$\hat{u}'_j\cdot\left\{\frac{\partial\hat{u}_i}{\partial t} + i\Delta_{in}k_m(\hat{u}_n\otimes\hat{u}_m) = -\nu k^2\hat{u}_i\right\}$$

$$\hat{u}_i\cdot\left\{\frac{\partial\hat{u}'_j}{\partial t} + i\Delta'_{jn}k'_m(\hat{u}_n\otimes\hat{u}_m)' = -\nu k'^2\hat{u}'_j\right\}$$

$\hat{u}'_j = \hat{u}_j(\vec{k}',t)$ and $\hat{u}_i = \hat{u}_i(\vec{k},t)$

$$\frac{\partial\hat{u}_i\hat{u}'_j}{\partial t} + i\left\{\Delta'_{jn}k'_m\hat{u}_i(\hat{u}_n\otimes\hat{u}_m)' + \Delta_{in}k_m\hat{u}'_j(\hat{u}_n\otimes\hat{u}_m)\right\} = -\nu(k^2 + k'^2)\hat{u}_i\hat{u}'_j$$

Recall $\overline{\hat{u}_i(\vec{k},t)\hat{u}_j(\vec{k}',t)} = \Phi_{ij}(\vec{k},t)\delta(\vec{k}+\vec{k}')$

$$\Rightarrow \quad \frac{\partial\Phi_{ij}}{\partial t} - T_{ij} = -2\nu k^2\Phi_{ij}$$

① $i\Delta'_{jn}k'_m\hat{u}_i(\hat{u}_n\otimes\hat{u}_m)' = -i\Delta_{jn}k_m\iint_{\vec{p}+\vec{q}+\vec{k}=0}\hat{u}_i(\vec{k})\hat{u}_n(\vec{p})\hat{u}_m(\vec{q})d\vec{p}d\vec{q}$

② $i\Delta_{in}k_m\hat{u}'_j(\hat{u}_n\otimes\hat{u}_m) = i\Delta_{in}k_m\iint_{\vec{p}+\vec{q}-\vec{k}=0}\hat{u}_j(-\vec{k})\hat{u}_n(\vec{p})\hat{u}_m(\vec{q})d\vec{p}d\vec{q}$

$$\Theta_{inm}(\vec{k}) \equiv \iint_{\vec{p}+\vec{q}+\vec{k}=0}\overline{\hat{u}_i(\vec{k})\hat{u}_n(\vec{p})\hat{u}_m(\vec{q})}d\vec{p}d\vec{q}$$

$\overline{\text{①} + \text{②}} = T_{ij}(\vec{k}) = ik_m\left(\Delta_{in}\Theta_{jnm}(-\vec{k}) - \Delta_{jn}\Theta_{inm}(\vec{k})\right)$

= energy transfer between different wave vectors and the given wave vector $\vec{k}$ (triad interaction)

8. Spectral Analysis of Homogeneous Turbulence

$$\frac{\partial \Phi_{ij}}{\partial t} = T_{ij} - 2\nu k^2 \Phi_{ij}$$

$$\tfrac{1}{2}\overline{u_i u_i} = \int \tfrac{1}{2}\Phi_{ii}(\vec{k},t)\,d\vec{k}$$

~ involve third-order correlations (closure problem)

~ at low waver numbers, T_{ij} dominates (energy cascade)

~ viscous dissipation mainly at large wave number

$$\bar{\varepsilon} = 2\nu\int k^2 \Phi_{ii}(\vec{k})\,d\vec{k} = 2\nu\int_0^\infty k^2 E(k)\,dk$$

~ a range of intermediate wave numbers, $L^{-1} << k << \eta^{-1}$ (inertial subrange)

~ T_{ii} = energy cascade $\int T_{ii}(\vec{k})\,d\vec{k} = 0$

8. Spectral Analysis of Homogeneous Turbulence

$$T_{ij} = ik_m\left(\Delta_{jn}\Theta_{inm}(\vec{k}) - \Delta_{in}\Theta_{jnm}(-\vec{k})\right) \Rightarrow T_{ii} = 2k_m\Delta_{in}\,\mathrm{Im}\,\Theta_{inm}(\vec{k})$$

$$\Theta_{inm}(\vec{k}) \equiv \iint_{\vec{p}+\vec{q}+\vec{k}=0} \overline{\hat{u}_i(\vec{k})\hat{u}_n(\vec{p})\hat{u}_m(\vec{q})}\,d\vec{p}\,d\vec{q}$$

Due to the symmetry: $T_{ii}(\vec{k}) = \iint_{\vec{p}+\vec{q}+\vec{k}=0} S(\vec{k};\vec{p},\vec{q})\,d\vec{p}\,d\vec{q}$

$$T_{ii} = \mathrm{Im}(k_m\Delta_{in} + k_n\Delta_{im})\Theta_{inm}(\vec{k})$$

$$T_{ii} = \iint_{\vec{p}+\vec{q}+\vec{k}=0} \mathrm{Im}\left\{(k_m\Delta_{in} + k_n\Delta_{im})\overline{\hat{u}_i(\vec{k})\hat{u}_n(\vec{p})\hat{u}_m(\vec{q})}\right\}d\vec{p}\,d\vec{q}$$

$$S(\vec{k};\vec{p},\vec{q}) \equiv \mathrm{Im}\left\{(k_m\Delta_{in} + k_n\Delta_{im})\overline{\hat{u}_i(\vec{k})\hat{u}_n(\vec{p})\hat{u}_m(\vec{q})}\right\}$$

= energy transfer rate into mode $\vec{k}$ through its interaction with modes $\vec{p}$ and $\vec{q}$, where $\vec{p}+\vec{q}+\vec{k}=0$

8. Spectral Analysis of Homogeneous Turbulence

$$S(\vec{k};\vec{p},\vec{q}) \equiv \mathrm{Im}\left\{(k_m\Delta_{in} + k_n\Delta_{im})\overline{\hat{u}_i(\vec{k})\hat{u}_n(\vec{p})\hat{u}_m(\vec{q})}\right\}$$

detailed conservation:

$$S(\vec{k};\vec{p},\vec{q}) + S(\vec{p};\vec{q},\vec{k}) + S(\vec{q};\vec{k},\vec{p}) = \mathbf{0} \qquad \boxed{\vec{p}+\vec{q}+\vec{k}=0}$$

global conservation:

$$\int T_{ii}(\vec{k})\,d\vec{k} = \int \iint_{\vec{p}+\vec{q}+\vec{k}=0} S(\vec{k};\vec{p},\vec{q})\,d\vec{p}\,d\vec{q}\;\;d\vec{k} = 0$$

<proof>

$$S(\vec{k};\vec{p},\vec{q}) \equiv \mathrm{Im}\left\{k_m\left(\delta_{in} - \cancel{\frac{k_i k_n}{k^2}}\right) + k_n\left(\delta_{im} - \cancel{\frac{k_i k_m}{k^2}}\right)\right\}\overline{\hat{u}_i(\vec{k})\hat{u}_n(\vec{p})\hat{u}_m(\vec{q})}$$

$$= \mathrm{Im}\left\{k_m\overline{\hat{u}_i(\vec{k})\hat{u}_i(\vec{p})\hat{u}_m(\vec{q})} + k_n\overline{\hat{u}_i(\vec{k})\hat{u}_n(\vec{p})\hat{u}_i(\vec{q})}\right\}$$

$$= \mathrm{Im}\left\{k_m\left(\overline{\hat{u}_i(\vec{k})\hat{u}_i(\vec{p})\hat{u}_m(\vec{q})} + \overline{\hat{u}_i(\vec{k})\hat{u}_m(\vec{p})\hat{u}_i(\vec{q})}\right)\right\}$$

8. Spectral Analysis of Homogeneous Turbulence

$$S(\vec{k};\vec{p},\vec{q}) + S(\vec{p};\vec{q},\vec{k}) + S(\vec{q};\vec{k},\vec{p})$$

$$= \mathrm{Im}\left\{k_m\left(\overline{\hat{u}_i(\vec{k})\hat{u}_i(\vec{p})\hat{u}_m(\vec{q})} + \overline{\hat{u}_i(\vec{k})\hat{u}_m(\vec{p})\hat{u}_i(\vec{q})}\right)\right\}$$

$$+ \mathrm{Im}\left\{p_m\left(\overline{\hat{u}_i(\vec{p})\hat{u}_i(\vec{q})\hat{u}_m(\vec{k})} + \overline{\hat{u}_i(\vec{p})\hat{u}_m(\vec{q})\hat{u}_i(\vec{k})}\right)\right\}$$

$$+ \mathrm{Im}\left\{q_m\left(\overline{\hat{u}_i(\vec{q})\hat{u}_i(\vec{k})\hat{u}_m(\vec{p})} + \overline{\hat{u}_i(\vec{q})\hat{u}_m(\vec{k})\hat{u}_i(\vec{p})}\right)\right\} \qquad \boxed{\vec{p}+\vec{q}+\vec{k}=0}$$

$$= \mathrm{Im}\left\{k_m\left(\overline{\hat{u}_i(\vec{k})\hat{u}_i(\vec{p})\hat{u}_m(\vec{q})} + \overline{\hat{u}_i(\vec{k})\hat{u}_m(\vec{p})\hat{u}_i(\vec{q})}\right)\right\}$$

$$+ \mathrm{Im}\left\{(-q_m - k_m)\left(\overline{\hat{u}_i(\vec{p})\hat{u}_i(\vec{q})\hat{u}_m(\vec{k})} + \overline{\hat{u}_i(\vec{p})\hat{u}_m(\vec{q})\hat{u}_i(\vec{k})}\right)\right\}$$

$$+ \mathrm{Im}\left\{(-p_m - k_m)\left(\overline{\hat{u}_i(\vec{q})\hat{u}_i(\vec{k})\hat{u}_m(\vec{p})} + \overline{\hat{u}_i(\vec{q})\hat{u}_m(\vec{k})\hat{u}_i(\vec{p})}\right)\right\}$$

8. Spectral Analysis of Homogeneous Turbulence

$$S(\vec{k};\vec{p},\vec{q})+S(\vec{p};\vec{q},\vec{k})+S(\vec{q};\vec{k},\vec{p})$$

$$=\mathrm{Im}\left\{k_m\left(\overline{\hat{u}_i(\vec{k})\hat{u}_i(\vec{p})\hat{u}_m(\vec{q})}+\overline{\hat{u}_i(\vec{k})\hat{u}_m(\vec{p})\hat{u}_i(\vec{q})}\right)\right\}$$

$$+\mathrm{Im}\left\{-q_m\overline{\hat{u}_i(\vec{p})\hat{u}_i(\vec{q})\hat{u}_m(\vec{k})}-k_m\overline{\hat{u}_i(\vec{p})\hat{u}_m(\vec{q})\hat{u}_i(\vec{k})}\right\}$$

$$+\mathrm{Im}\left\{-k_m\overline{\hat{u}_i(\vec{q})\hat{u}_i(\vec{k})\hat{u}_m(\vec{p})}-p_m\overline{\hat{u}_i(\vec{q})\hat{u}_m(\vec{k})\hat{u}_i(\vec{p})}\right\}$$

$$=\mathrm{Im}\left\{-q_m\overline{\hat{u}_i(\vec{p})\hat{u}_i(\vec{q})\hat{u}_m(\vec{k})}-p_m\overline{\hat{u}_i(\vec{q})\hat{u}_m(\vec{k})\hat{u}_i(\vec{p})}\right\}$$

$$=\mathrm{Im}\left\{-(q_m+p_m)\overline{\hat{u}_i(\vec{p})\hat{u}_i(\vec{q})\hat{u}_m(\vec{k})}\right\} \qquad \boxed{\vec{p}+\vec{q}+\vec{k}=0}$$

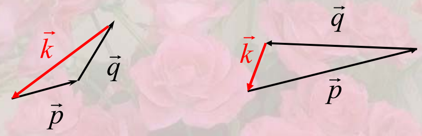
$$=\mathrm{Im}\left\{k_m\overline{\hat{u}_i(\vec{p})\hat{u}_i(\vec{q})\hat{u}_m(\vec{k})}\right\}$$

= 0

8. Spectral Analysis of Homogeneous Turbulence

$$S(\vec{k};\vec{p},\vec{q})+S(\vec{p};\vec{q},\vec{k})+S(\vec{q};\vec{k},\vec{p})=0$$

$$\vec{p}+\vec{q}+\vec{k}=0$$



~ Energy is redistributed but conserved. ~

8. Spectral Analysis of Homogeneous Turbulence

$$\int_0^{2\pi}\int_0^{\pi}\frac{1}{2}\left\{\frac{\partial\Phi_{ii}(\vec{k})}{\partial t}=T_{ii}(\vec{k})-2\nu k^2\Phi_{ii}(\vec{k})\right\}k^2\sin\theta\, d\theta\, d\phi$$

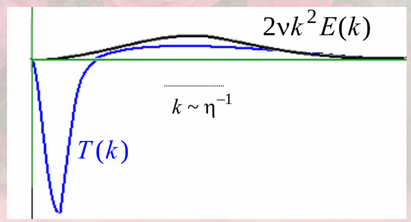
~ evolution equation of energy of mode $\vec{k}$

$$\frac{\partial E(k)}{\partial t}=T(k)-2\nu k^2E(k)$$

~ evolution equation of energy contained within modes of wave number k (eddies of length scale k^{-1})

$$\bar{\varepsilon}=2\nu\int_0^{\infty}k^2E(k)dk$$

$$\int_0^{\infty}T(k)dk=0$$



8. Spectral Analysis of Homogeneous Turbulence

$\Pi(k)\equiv-\int_0^k T(k')dk'$ **= the energy transfer rate from large eddies of scales < k^{-1} to small eddies of scales > k^{-1}**

energy-containing eddies:

$T(k)<0$: energy is extracted to smaller eddies

dissipation eddies:

$T(k)>0$: energy is extracted from larger eddies

inertial eddies:

$T(k)=0$ or $\Pi(k)=\text{constant}$

Energy is transferred into and out of the modes at a same rate.

8. In Physical Space

$$u'_j\left\{\frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_m}(u_iu_m)=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_m\partial x_m}\right\}$$

$$u_i\left\{\frac{\partial u'_j}{\partial t}+\frac{\partial}{\partial x'_m}(u'_ju'_m)=-\frac{1}{\rho}\frac{\partial p'}{\partial x'_j}+\nu\frac{\partial^2 u'_j}{\partial x'_m\partial x'_m}\right\}$$

$u_i\equiv u_i(\vec{x},t)$ and $u'_j\equiv u_j(\vec{x}',t)$

$\Downarrow$

$\vec{r}\equiv\vec{x}-\vec{x}'$

$$\frac{\partial}{\partial r_m}=\frac{\partial}{\partial x_m}=-\frac{\partial}{\partial x'_m}$$

$$\frac{\partial R_{ij}}{\partial t}=\Gamma_{ij}+\Pi_{ij}+2\nu\frac{\partial^2 R_{ij}}{\partial r_m\partial r_m}$$

$$\Gamma_{ij}(\vec{r},t)\equiv\frac{\partial}{\partial r_m}\overline{u_i(\vec{x},t)u_m(\vec{x},t)u_j(\vec{x}',t)-u_i(\vec{x},t)u_m(\vec{x}',t)u_j(\vec{x}',t)}$$

$$\Pi_{ij}(\vec{r},t)\equiv\frac{1}{\rho}\left(\frac{\partial}{\partial r_i}\overline{u_j(\vec{x}',t)p(\vec{x},t)}-\frac{\partial}{\partial r_j}\overline{u_i(\vec{x},t)p(\vec{x}',t)}\right)$$

8. In Physical Space

$$\frac{\partial R_{ii}}{\partial t}=\Gamma_{ii}+\Pi_{ii}+2\nu\frac{\partial^2 R_{ii}}{\partial r_m\partial r_m}$$

where

$$\Pi_{ii}(\vec{r})\equiv\frac{1}{\rho}\left(\frac{\partial}{\partial r_i}\overline{u'_ip}-\frac{\partial}{\partial r_i}\overline{u_ip'}\right)=-\frac{1}{\rho}\left(\frac{\partial\overline{u'_ip}}{\partial x'_i}+\frac{\partial\overline{u_ip'}}{\partial x_i}\right)=0$$

Fourier transform:

$$\frac{\partial\Phi_{ii}}{\partial t}=T_{ii}-2\nu k^2\Phi_{ii}$$

Pressure-velocity correlation does not change energy distribution among different wave vectors, but in directions.

8. Isotropic Turbulence

- **double velocity correlations**

$$R_{ij}(\vec{r})=\overline{u_i(\vec{x})u_j(\vec{x}+\vec{r})}=q^2\left\{\frac{f(r)-g(r)}{r^2}r_ir_j+g(r)\delta_{ij}\right\}$$

$$q^2f(r)\equiv\overline{u_{\parallel}(\vec{x})u_{\parallel}(\vec{x}+\vec{r})}\quad,\quad u_{\parallel}=\vec{u}\cdot\frac{\vec{r}}{r}$$

longitudinal correlation coefficient

$$q^2g(r)\equiv\overline{u_{\perp}(\vec{x})u_{\perp}(\vec{x}+\vec{r})}$$

transverse correlation coefficient

Incompressibility:

$$\frac{\partial R_{ij}}{\partial r_i}=0\quad\text{or}\quad f(r)+\frac{r}{2}\frac{\partial f}{\partial r}=g(r)$$

8. Isotropic Turbulence

- **triple velocity correlations**

$$S_{ij,m}(\vec{r})=\overline{u_i(\vec{x})u_j(\vec{x})u_m(\vec{x}+\vec{r})}$$

$$=q^3\left\{(k-h-2w)\frac{r_ir_jr_m}{r^3}+\delta_{ij}h\frac{r_m}{r}+w\left(\delta_{im}\frac{r_j}{r}+\delta_{jm}\frac{r_i}{r}\right)\right\}$$

$$q^3k(r)=\overline{u_{\parallel}(\vec{x})u_{\parallel}(\vec{x})u_{\parallel}(\vec{x}+\vec{r})}$$

$$q^3h(r)=\overline{u_{\perp}(\vec{x})u_{\perp}(\vec{x})u_{\parallel}(\vec{x}+\vec{r})}$$

$$q^3w(r)=\overline{u_{\perp}(\vec{x})u_{\parallel}(\vec{x})u_{\perp}(\vec{x}+\vec{r})}$$

Incompressibility:

$$\frac{\partial S_{ij,m}}{\partial r_m}=0\quad\Rightarrow\quad w=\frac{1}{4r}\frac{\partial}{\partial r}\left(r^2k\right)\quad\text{and}\quad h=-\tfrac{1}{2}k$$

8. Isotropic Turbulence

$$\frac{\partial R_{ii}}{\partial t} = \Gamma_{ii} + 2\nu \frac{\partial^2 R_{ii}}{\partial r_m \partial r_m}$$

$$\Downarrow$$

$$\frac{\partial}{\partial t}\left(q^2 f\right) = q^3\left(\frac{\partial k}{\partial r} + \frac{4k}{r}\right) + 2\nu q^2 \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial f}{\partial r}\right)$$

Karman-Howarth equation

~ incompressibility isotropic homogeneous turbulence ~

• Karman Th. von and Howarth L., “On the statistical theory of isotropic turbulence”, Proc. R. Soc. London Ser. A **164**, 192 (1938).

•Huang M.J. and Leonard A., “Power-law decay of homogeneous turbulence at low Reynolds numbers”, Phys. Fluids **6**, 3765 (1994).