2. Root Searching --- solutions of equations in one variable

Given: $f(x) \in C(X)$

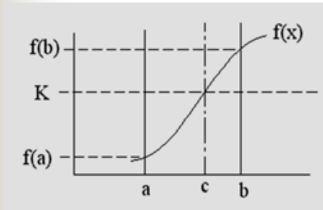
Find: x_0 such that $f(x_0) = 0$

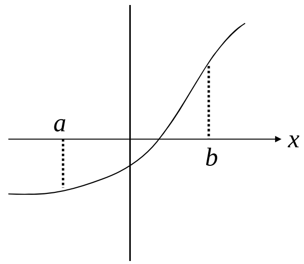
Methods: Bisection; Fixed-point; Newton-Raphson; Secant

§ Bisection Method

Step 1: look for a domain (a, b) where $f(a)*f(b) < 0$

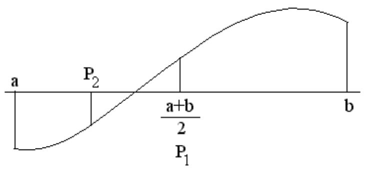
Intermediate value theorem: there exists at least one root in (a, b)





§ Bisection Method

Step 2: check either $f\left(\frac{a+b}{2}\right) \cdot f(a) < 0$ or $f\left(\frac{a+b}{2}\right) \cdot f(b) < 0$



$$\text{new } (a,b) = \begin{cases} \left(a, \frac{a+b}{2}\right) & \text{if } f\left(\frac{a+b}{2}\right) \cdot f(a) < 0 \\ \left(\frac{a+b}{2}, b\right) & \text{otherwise} \end{cases}$$

§ Bisection Method

Step 3: repeat Step 2 until $\left| f\left(\frac{a+b}{2}\right) \right| < \varepsilon$ (error tolerance)

Theorem: If $f(x) \in C[a,b]$, then $|P_n - x_0| \le \frac{b-a}{2^n}$,

where P_n is midpoint after n iterations.

§ Fixed-Point Method

Def : Given $g(x)$, a fixed point x_0 of $g(x)$ is a point at which $g(x_0) = x_0$

Let $g(x) \equiv x + f(x)$.

$\Rightarrow$ A root of $f(x)$ is a fixed point of $g(x)$.

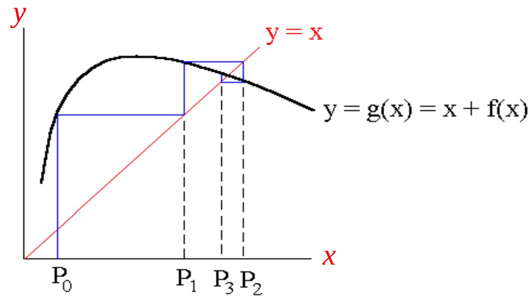
STEP1 : Take an initial guess P_0

STEP2: $P_1 = g(P_0) = P_0 + f(P_0)$

STEP2 : $P_n = g(P_{n-1}) = P_{n-1} + f(P_{n-1})$

STEP3: Stop the iteration and get $x_0 \approx P_n$ whenever $|P_n - g(P_n)| < \varepsilon$

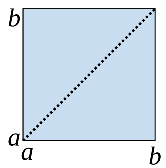
STEP2 : $P_n = g(P_{n-1}) = P_{n-1} + f(P_{n-1})$



§ Fixed-Point Theorems

Theorem: If $g(x) \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$,

(existence) then $g(x)$ has a fixed point in $[a,b]$.



(uniqueness): In addition, $g'(x)$ exists on $[a,b]$ and $\exists k,\ 0 < k < 1$,

and $|g'(x)| \le k < 1$ for all $x \in (a,b)$,

then the fixed point in $[a,b]$ is unique.

(convergence): $\lim_{n\to\infty} |P_n - x_0| \le \lim_{n\to\infty} k^n |P_0 - x_0| = 0 \quad (\because k < 1)$

Def. (convergence): An infinite sequence $\{P_n\}_{n=0}^{\infty}$ is said to converge to a number x_0 if given any $\varepsilon > 0$, $\exists$ a positive number $N(\varepsilon)$ such that for all $n > N(\varepsilon)$, $|P_n - x_0| < \varepsilon$. Write $\lim_{n\to\infty} P_n = x_0$.

e.g. $\lim_{n\to\infty} \frac{n+1}{n} = 1$

choose $N = 1/\varepsilon$

$$|P_n - x_0| = \left|\frac{n+1}{n} - 1\right| = \frac{1}{n} < \frac{1}{N} = \varepsilon$$

Def. (rate of convergence):

Suppose $\{P_n\}_{n=0}^{\infty}$ is a sequence that converges to x_0.

If $\exists\lambda$, $\alpha>0$ such that $\lim_{n\to\infty}\frac{|P_{n+1}-x_0|}{|P_n-x_0|^{\alpha}}=\lambda$

then the sequence is said to converge to x_0 of order α with an asymptotic error constant λ.

- The higher the order α is, the faster the sequence converges.

e.g. $|P_n-x_0|\sim O(10^{-2})$ for some sufficiently large n

$$|P_{n+1}-x_0|\sim|P_n-x_0|^{\alpha}\sim \begin{cases} O(10^{-2}) & \text{if } \alpha=1 \\ O(10^{-4}) & \text{if } \alpha=2 \end{cases}$$

- linear convergence if $\alpha = 1$ (e.g. Bisection method)
- quadratic convergence if $\alpha=2$ (e.g. Newton-Raphson method)
- superliner if $\alpha=1$ and $\lambda = 0$

Theorem (convergence of fixed-point method)

Let $g(x)\in C^r[a,b]$, $x_0=g(x_0)$, $x_0\in[a,b]$.

If $g'(x_0)=g''(x_0)=\cdots=g^{(r-1)}(x_0)=0$ but $g^{(r)}(x_0)\neq 0$,

then there exists $\delta>0$ such that for $P_0\in[x_0-\delta,x_0+\delta]$

(*i*) $\{P_n\}_{n=0}^{\infty}$ converges to x_0 of order r ;

(*ii*) If $g^{(r)}(x)$ is bounded by M on $[x_0-\delta,x_0+\delta]$, then

$$|P_{n+1}-x_0|<\frac{M}{r!}|P_n-x_0|^r$$

$$P_{n+1}=g(P_n)=g(x_0+P_n-x_0)=g(x_0+e_n)$$

Lagrange remainder

$$=g(x_0)+e_n g'(x_0)+\frac{e_n^2}{2!}g''(x_0)+\cdots+\frac{e_n^{r-1}}{(r-1)!}g^{(r-1)}(x_0)+\frac{e_n^r}{r!}g^{(r)}(\xi)$$

If $g'(x_0)=g''(x_0)=\cdots=g^{(r-1)}(x_0)=0$ but $g^{(r)}(x_0)\neq 0$,

$$=x_0+\frac{e_n^r}{r!}g^{(r)}(\xi)$$

$$\lim_{n\to\infty}\frac{|P_{n+1}-x_0|}{|P_n-x_0|^{\alpha}}=\lambda$$

$$P_{n+1}-x_0=+\frac{e_n^r}{r!}g^{(r)}(\xi)$$

$$\frac{|e_{n+1}|}{|e_n^r|}=\frac{|P_{n+1}-x_0|}{|P_n-x_0|^r}=\frac{1}{r!}|g^{(r)}(\xi)|<\frac{M}{r!}$$

Theorem (convergence of fixed-point method)

Let $g(x)\in C^r[a,b]$, $x_0=g(x_0)$, $x_0\in[a,b]$.

If $g'(x_0)=g''(x_0)=\cdots=g^{(r-1)}(x_0)=0$ but $g^{(r)}(x_0)\neq 0$,

then there exists δ>0 such that for $P_0\in[x_0-\delta,x_0+\delta]$

$$|P_{n+1}-x_0|<\frac{M}{r!}|P_n-x_0|^r$$

If $g'(x_0)\neq 0$, the fixed-point method is a 1st order method $(r=1)$.

If $|g'(x)|\leq k$, then $|P_{n+1}-x_0|<k|P_n-x_0|$.

$$|P_n-x_0|<k^n|P_0-x_0|$$

Theorem (convergence of fixed-point method)

Let $g(x)\in C^r[a,b]$, $x_0=g(x_0)$, $x_0\in[a,b]$.

If $g'(x_0)=g''(x_0)=\cdots=g^{(r-1)}(x_0)=0$ but $g^{(r)}(x_0)\neq 0$,

then there exists δ>0 such that for $P_0\in[x_0-\delta,x_0+\delta]$

$$|P_{n+1}-x_0|<\frac{M}{r!}|P_n-x_0|^r$$

In reality, we don't know what r is.

We don't know how large δ can be.

We don't know where x_0 is.

The fixed-point method is at least of order one but a convergence can be obtained only by try-and-error.

§ Newton-Raphson Method (quadratic order)

Fixed Point Method: $g(x)\equiv x+f(x)$

$$P_{n+1}=g(P_n)$$

We don't know what *r* is.

Can we construct a special function $g(x)$ which $g'(x_0)=0$ for sure even without knowing x_0. Therefore, from the above theorem, a method of order two is obtained.

Let $g(x)=x-\phi(x)f(x)$.

$$\text{Thus}\begin{cases} g(x_0)=x_0-\phi(x_0)f(x_0)=x_0-\phi(x_0)\cdot 0=x_0 \\ g'(x_0)=1-\phi'(x_0)f(x_0)-\phi(x_0)f'(x_0)=1-\phi(x_0)f'(x_0)\end{cases}$$

desired $0=g'(x_0)=1-\phi(x_0)f'(x_0)$

Choose $\phi(x)=\dfrac{1}{f'(x)}$ and thus $g(x)=x-\phi(x)f(x)=x-\dfrac{f(x)}{f'(x)}$

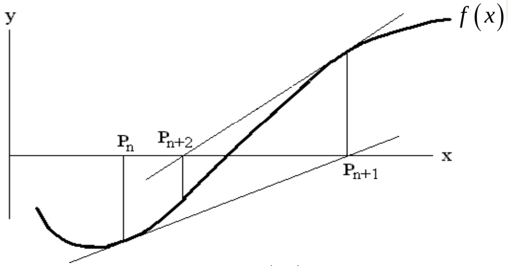
$\ni g(x_0)=x_0$ and $g'(x_0)=0$

§ Newton-Raphson Method (quadratic order)

Step 1: Take an initial guess P_0.

Step 2: $P_{n+1}=g(P_n)=P_n-\dfrac{f(P_n)}{f'(P_n)}$

comared with the fixed point method: $P_{n+1}=g(P_n)=P_n+f(P_n)$



the tangent line: $\dfrac{y-f(P_n)}{x-P_n}=f'(P_n)$

x-intersection (y=0): $x=P_n-\dfrac{f(P_n)}{f'(P_n)}$

Problem arises when $f'(P_n)=0$ during the iteration.

§ Newton-Raphson Method (quadratic order)

Theorem: Let $f(x)\in C^2[a,b]$, $f(x_0)=0$ and $f'(x_0)\neq 0$, then there exists a $\delta>0$ such that $\{P_n\}_{n=0}^{\infty}$ converges to x_0 for any initial guess $P_0\in[x_0-\delta, x_0+\delta]$ at least quadratically.

Question.1: We don't know x_0. We don't know δ either.

So, where is $[x_0-\delta, x_0-\delta]$?

We need to start the iteration with a good initial guess.

Solution: use low-order methods first such as bisection, fixed-point method, etc, followed by the Newton-Raphson method.

Question.2: $f'(x_0)\neq 0$ and $f'(P_n)\neq 0$ required

Theorem: $f(x)\in C^r[a,b]$ has a zero, x_0, of multiplicity r if and only if

$$f(x_0)=f'(x_0)=\cdots=f^{(r-1)}(x_0)=0 \text{ but } f^{(r)}(x_0)\neq 0.$$

Solution: Write $f(x)=(x-x_0)^r q(x)$ where $q(x_0)\neq 0$.

let $\mu(x)\equiv f(x)/f'(x)$

Then $\mu(x_0)=0$ and $\mu'(x_0)=1/r\neq 0$.

Look for a root of $f(x)$=0? ⇨ look for a root of $\mu(x)$=0 !

⇨ quadratic convergence ensured!

<Proof>: Write $f(x)=(x-x_0)^r q(x)$ where $q(x_0)\neq 0$.

$\mu(x)\equiv f(x)/f'(x)$ Then $\mu(x_0)=0$ and $\mu'(x_0)=1/r\neq 0$.

$$\mu(x)=\frac{(x-x_0)^r q(x)}{(x-x_0)^r q'(x)+r(x-x_0)^{r-1}q(x)}=\frac{\cancel{(x-x_0)}q(x)}{(x-x_0)q'(x)+rq(x)}$$

$$\mu'(x)=\left\{\begin{matrix}\left[\cancel{(x-x_0)q'(x)}+q(x)\right]\left[\cancel{(x-x_0)q'(x)}+rq(x)\right]\\ -\left[\cancel{(x-x_0)q(x)}\right]\left[\cancel{(x-x_0)q''(x)+(r+1)q'(x)}\right]\end{matrix}\right\}\Big/\left[\cancel{(x-x_0)q'(x)}+rq(x)\right]^2$$

Thus $\mu(x_0)=0$ and $\mu'(x_0)=1/r\neq 0$.

Apply Newton-Raphson method to $\mu(x)$:

$$g(x) = x - \frac{\mu(x)}{\mu'(x)} = x - \frac{f/f'}{(f/f')'} = x - \frac{f(x)f'(x)}{(f'(x))^2 - f(x)f''(x)}$$

Question.3: increasing rounding errors as $P_n \to x_0$

because $|f(P_n)|$ and $|f'(P_n)|$ and possibly also $|f''(P_n)|$ are very small.

§ Secant Method (superlinear order)

Newton-Raphson Method: $g(x) = x - \frac{f(x)}{f'(x)}$ or $g(x) = x - \frac{\mu(x)}{\mu'(x)}$

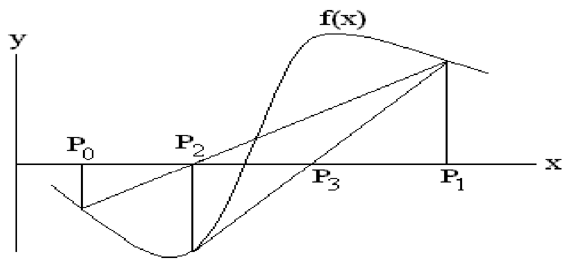
We want to avoid computations of $f'(P_n)$ and $f''(P_n)$

Replace $f'(P_n)$ by $\frac{f(P_n) - f(P_{n-1})}{P_n - P_{n-1}}$

Step 1: Take two initial guesses P_0 and P_1

Step 2: let $P_{n+1} = g(P_n) = P_n - f(P_n) \cdot \frac{P_n - P_{n-1}}{f(P_n) - f(P_{n-1})}$

Secant method: $P_{n+1} = P_n - f(P_n) \cdot \frac{P_n - P_{n-1}}{f(P_n) - f(P_{n-1})}$



Theorem: Let $f(x) \in C^2[a,b]$ and $\exists x_0 \in [a,b]$ such that $f(x_0) = 0$, $f'(x_0) \neq 0$, and $f''(x_0) \neq 0$. Then the Secant iteration converges to x_0 of order $(1+\sqrt{5})/2$ (superlinear).

<Proof> Define $e_{n+1} = P_{n+1} - x_0 =$ error after n+1 iterations

$$P_{n+1} - x_0 = \left\{ P_n - f(P_n) \cdot \frac{P_n - P_{n-1}}{f(P_n) - f(P_{n-1})} \right\} - x_0$$

$$= e_n - f(P_n) \cdot \frac{e_n - e_{n-1}}{f(P_n) - f(P_{n-1})}$$

Use Taylor's series theorem:

$$f(P_n)=f(x_0+P_n-x_0)=f(x_0+e_n)$$

$$\approx f(x_0)+f'(x_0)e_n+\frac{1}{2!}f''(x_0)e_n^2+\cdots$$

$$f(P_{n-1})\approx f(x_0)+f'(x_0)e_{n-1}+\frac{1}{2!}f''(x_0)e_{n-1}^2+\cdots$$

$$f(P_n)-f(P_{n-1})\approx f'(x_0)(e_n-e_{n-1})+\frac{1}{2}f''(x_0)\left(e_n^2-e_{n-1}^2\right)+\cdots$$

$$\frac{e_n-e_{n-1}}{f(P_n)-f(P_{n-1})}=\frac{1}{f'(x_0)+\frac{1}{2}f''(x_0)(e_n+e_{n-1})+\cdots}$$

$$=\frac{1}{f'(x_0)}\left(1+\frac{1}{2}\frac{f''(x_0)}{f'(x_0)}(e_n+e_{n-1})+\cdots\right)^{-1}$$

$$\frac{e_n-e_{n-1}}{f(P_n)-f(P_{n-1})}=\frac{1}{f'(x_0)}\left(1-\frac{1}{2}\frac{f''(x_0)}{f'(x_0)}(e_n+e_{n-1})+\cdots\right)$$

$$f(P_n)=f'(x_0)e_n+\frac{1}{2!}f''(x_0)e_n^2+\cdots=f'(x_0)e_n\left(1+\frac{1}{2}\frac{f''(x_0)}{f'(x_0)}e_n+\cdots\right)$$

$$e_{n+1}=e_n-f(P_n)\cdot\frac{e_n-e_{n-1}}{f(P_n)-f(P_{n-1})}$$

$$=e_n-e_n\left(1+\frac{1}{2}\frac{f''(x_0)}{f'(x_0)}e_n+\cdots\right)\left(1-\frac{1}{2}\frac{f''(x_0)}{f'(x_0)}(e_n+e_{n-1})+\cdots\right)$$

$$=e_n-e_n\left(1+\frac{1}{2}\frac{f''(x_0)}{f'(x_0)}e_n-\frac{1}{2}\frac{f''(x_0)}{f'(x_0)}(e_n+e_{n-1})+\cdots\right)\quad\approx\frac{1}{2}\frac{f''(x_0)}{f'(x_0)}e_ne_{n-1}$$

$$e_{n+1}\approx\frac{1}{2}\frac{f''(x_0)}{f'(x_0)}e_ne_{n-1}$$

~ for sufficient small e_n and e_{n-1}

~ for sufficient large n

$$e_{n+1}=Ce_ne_{n-1}$$

Suppose the Secant method has a convergence order of α, that is

$$\lim_{n\to\infty}\frac{|P_{n+1}-x_0|}{|P_n-x_0|^{\alpha}}=\lambda=\lim_{n\to\infty}\frac{|e_{n+1}|}{|e_n|^{\alpha}}$$

For sufficiently large n, $\quad |e_{n+1}|\approx\lambda|e_n|^{\alpha}$

Substitute into $e_{n+1}=Ce_ne_{n-1}$

$$\lambda|e_n|^{\alpha}=|C|\cdot|e_n|\cdot\left(|e_n|/\lambda\right)^{1/\alpha}$$

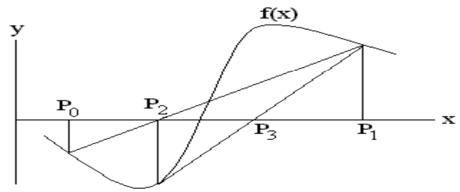
$$|e_n|^{\alpha}\sim|e_n|^{1+1/\alpha}$$

$$\alpha=1+1/\alpha\quad\Rightarrow\quad\alpha=\frac{1+\sqrt{5}}{2}\approx1.62$$

Summary:

- bisection method $\alpha=1$ ensure $x_0 \in (P_n, P_{n+1})$
- fixed-point method $\alpha = 1$
- Newton-Raphson method $\alpha = 2$
- Secant method $\alpha = 1.62$

§ Secant Method + Bisection Method

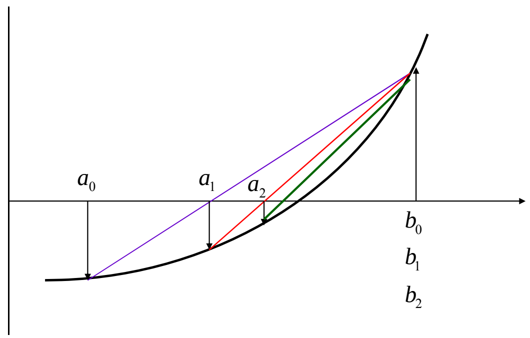


Step 1: Take two initial guesses a_0 and b_0 $\ni f(P_0)\cdot f(P_1) < 0$.

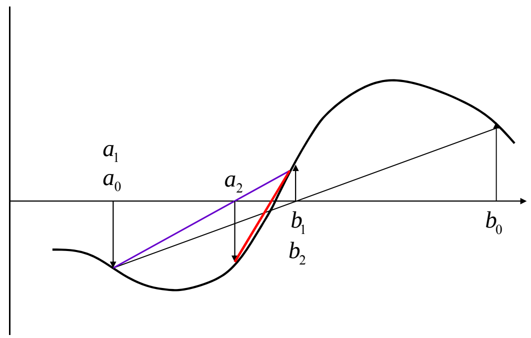
Step 2: let $P_{n+1} = b_n - f(b_n)\cdot \dfrac{b_n - a_n}{f(b_n) - f(a_n)}$

Step 3: Choose $(a_{n+1}, b_{n+1}) = \begin{cases} (a_n, P_{n+1}) & \text{if } f(a_n)\cdot f(P_{n+1}) < 0 \\ (P_{n+1}, b_n) & \text{if } f(b_n)\cdot f(P_{n+1}) < 0 \end{cases}$

$x_0 \in (a_{n+1}, b_{n+1})$ always!



$x_0 \in (a_{n+1}, b_{n+1})$ always!



§ Aitken's Δ^2 method --- speeding convergence

Given: $\{P_n\}_{n=0}^{\infty}$ is a sequence converging to x_0

Desired: A new sequence converges to x_0 at a faster speed.

Idea: for sufficiently large n: $\dfrac{P_{n+1}-x_0}{P_n-x_0} \approx \dfrac{P_{n+2}-x_0}{P_{n+1}-x_0}$

$$x_0 \approx P_n - \frac{(P_{n+1}-P_n)^2}{(P_{n+2}-2P_{n+1}+P_n)}$$

$$\hat{P}_n \equiv P_n - \frac{(P_{n+1}-P_n)^2}{(P_{n+2}-2P_{n+1}+P_n)} = P_n - \frac{(\Delta P_n)^2}{\Delta^2 P_n}$$

where

$$\Delta P_n \equiv P_{n+1} - P_n$$

$$\Delta^2 P_n \equiv \Delta(\Delta P_n) = \Delta(P_{n+1}-P_n) = P_{n+2} - 2P_{n+1} + P_n$$

$$\Delta^k P_n \equiv \Delta(\Delta^{k-1} P_n)$$

Theorem: Let $\lim\limits_{n\to\infty} \dfrac{P_{n+1}-x_0}{P_n-x_0} = \lambda$, $|\lambda| < 1$, and $P_n - x_0 \neq 0$ for all n.

Then $\{\hat{P}_n\}_{n=0}^{\infty}$ converges to x_0 faster than $\{P_n\}_{n=0}^{\infty}$ in the sense that

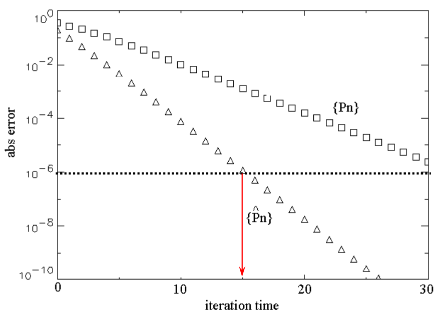
$$\lim_{n\to\infty} \frac{\hat{P}_n - x_0}{P_n - x_0} = 0$$

§ Aitken's Δ^2 method --- example

choose $f(x) = \sin x - x$ -

Searching by the fixed-p

with an initial guess P_0=(



$$\hat{P}_n \equiv P_n - \frac{(P_{n+1}-P_n)^2}{(P_{n+2}-2P_{n+1}+P_n)}$$

$$\begin{array}{ccccccccc} P_0 & \to & P_1 & \to & P_2 & \to & P_3 & \to & P_4 \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & \hat{P}_0 & & \hat{P}_1 & & \hat{P}_2 \end{array}$$

§ Application to Zeros of Polynomials

Given: $n \geq 1$, $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

Find: zeros of $P_n(x)$

Apply Newton-Raphson method ⇨ need $P_n(x)$ and $P_n'(x)$

direct computation for $P_n(c)$: $n + (n-1) + \cdots + 1 = \dfrac{n(n+1)}{2}$ ⊗

n ⊕

Theorem (Horner's method): $P_n(c)$ desired

(i) $b_n = a_n$ and $b_k = a_k + b_{k+1} * c$ for $k = n-1, \cdots, 1, 0$

need n⊗ and n⊕

(ii) Define $Q(x) = b_n x^{n-1} + b_{n-1}x^{n-2} + \cdots + b_2 x + b_1$

Then $P_n(x) = (x-c)Q(x) + b_0$

$\Rightarrow P_n(c) = b_0$ and $P_n'(c) = Q(c)$

example: Given $P_4(x) = 2x^4 - 3x^2 + 3x - 4$ Find $P_4(-2)$ and $P_4'(-2)$

$b_n = a_n$

$b_k = a_k + b_{k+1} * c$

	a_4	a_3	a_2	a_1	a_0
	2	0	−3	3	−4
−2		−4	8	−10	14
	2	−4	5	−7	10
	b_4	b_3	b_2	b_1	b_0
−2		−4	16	−42	
	2	−8	21	−49	

§ Műller's method for complex roots with real initial guesses

Step1: Take three initial complex or real guesses P_0, P_1, P_2

Step2: Construct a quadratic polynomial passing P_0, P_1, P_2

c.f. Secant method: a line passing P_0 and P_1

Step3: P_{n+1} = the zero of the quadratic polynomial passing P_n, P_{n-1}, P_{n-2}

Alternatively, when complex roots of polynomials $P_n(z) = 0$ with all reall coefficients are concerned:

- If $z = a+ib$ is a complex root, so is $z = a-ib$.

$P_n(z)$ has a factor of $(z-a-ib)(z-a+ib) = z^2 - 2az + a^2 + b^2$

Searching for a complex root?

Searching for $\alpha,\beta \in R$ such that $P_2(z) = z^2 - \alpha z - \beta$ is a factor of $P_n(z)$!

Theorem: Given any $P_n(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$

and $z^2 - \alpha z - \beta$

We write $P_n(z) = (z^2 - \alpha z - \beta)Q_{n-2}(z) + R_1(z)$

$Q_{n-2}(z) = b_n z^{n-2} + b_{n-1}z^{n-3} + \cdots + b_3 z + b_2$

$R_1(z) = b_1(z-\alpha) + b_0$

$b_{n+2} = b_{n+1} = 0$

$b_k = a_k + \alpha b_{k+1} + \beta b_{k+2}$ for $k = n, n-1, \cdots, 1, 0$

Wanted: Find a pair of (α,β) such that $b_0 = b_1 = 0$.

§ Bairstow's method

Step 1: Take one initial guess of real (α_0,β_0)

Step 2: Compute $\{b_k\}_{k=0}^{n+2}$ using (α_0,β_0). Check if $b_0=0$ and $b_1=0$

Step 3: Compute $\{c_k\}_{k=0}^{n+1}$

$$\begin{cases} c_{n+1}=c_n=0 \\ c_k=b_{k+1}+\alpha_i c_{k+1}+\beta_i c_{k+2} \quad \text{for } k=n-1,n-2,\cdots,1,0 \end{cases}$$

Step 4: Compute $J=c_0c_2-c_1^2$ and

$$\begin{cases} \alpha_{i+1}=\alpha_i+(c_1b_1-c_2b_0)/J \\ \beta_{i+1}=\beta_i+(c_1b_0-c_0b_1)/J \end{cases}$$

Given (α_i,β_i)

$$P_n(z)=(z^2-\alpha_i z-\beta_i)Q_{n-2}(z)+b_1(z-\alpha_i)+b_0$$

$$\Rightarrow \begin{cases} b_0=b_0(\alpha_i,\beta_i) \\ b_1=b_1(\alpha_i,\beta_i) \end{cases}$$

$b_0\neq 0$ or $b_1\neq 0$: need a new guess

Let $\begin{cases} \alpha_{i+1}=\alpha_i+\delta\alpha \\ \beta_{i+1}=\beta_i+\delta\beta \end{cases}$. We expect

$$\begin{cases} b_0(\alpha_{i+1},\beta_{i+1})=b_0(\alpha_i+\delta\alpha,\beta_i+\delta\beta)=0 \\ b_1(\alpha_{i+1},\beta_{i+1})=b_1(\alpha_i+\delta\alpha,\beta_i+\delta\beta)=0 \end{cases}$$

$$\begin{cases} b_0(\alpha_i+\delta\alpha,\beta_i+\delta\beta)\approx b_0(\alpha_i,\beta_i)+\delta\alpha\dfrac{\partial b_0}{\partial\alpha}(\alpha_i,\beta_i)+\delta\beta\dfrac{\partial b_0}{\partial\beta}(\alpha_i,\beta_i)=0 \\ b_1(\alpha_i+\delta\alpha,\beta_i+\delta\beta)\approx b_1(\alpha_i,\beta_i)+\delta\alpha\dfrac{\partial b_1}{\partial\alpha}(\alpha_i,\beta_i)+\delta\beta\dfrac{\partial b_1}{\partial\beta}(\alpha_i,\beta_i)=0 \end{cases}$$

$$c_k\equiv\frac{\partial b_k}{\partial\alpha}=\begin{cases} 0 \quad \text{for } k=n+1,n+2 \\ \dfrac{\partial}{\partial\alpha}(a_k+\alpha b_{k+1}+\beta b_{k+2}) \quad ,k=0,1,2,\cdots,n \end{cases}$$

$$=b_{k+1}+\alpha\frac{\partial b_{k+1}}{\partial\alpha}+\beta\frac{\partial b_{k+2}}{\partial\alpha} \quad =b_{k+1}+\alpha c_{k+1}+\beta c_{k+2}$$

$b_{n+2}=b_{n+1}=0$

$b_k=a_k+\alpha b_{k+1}+\beta b_{k+2}$ for $k=n,n-1,\cdots,1,0$

$$d_k\equiv\frac{\partial b_{k-1}}{\partial\beta}=\begin{cases} 0 \quad \text{for } k=n+2 \\ \dfrac{\partial}{\partial\beta}(a_{k-1}+\alpha b_k+\beta b_{k+1}) \quad ,k=n+1,n,\cdots,1 \end{cases}$$

$$=\alpha\frac{\partial b_k}{\partial\beta}+\beta\frac{\partial b_{k+1}}{\partial\beta}+b_{k+1} \quad =\alpha d_{k+1}+\beta d_{k+2}+b_{k+1}$$

$$d_{n+1}=\frac{\partial b_n}{\partial\beta}=\frac{\partial}{\partial\beta}(a_n+\alpha b_{n+1}+\beta b_{n+2})=0$$

$b_{n+2}=b_{n+1}=0$

$b_k=a_k+\alpha b_{k+1}+\beta b_{k+2}$ for $k=n,n-1,\cdots,1,0$

Summary:

$$c_k \equiv \frac{\partial b_k}{\partial \alpha} = \begin{cases} 0 & \text{for } k = n+1, n+2 \\ b_{k+1} + \alpha c_{k+1} + \beta c_{k+2} & , k = n, n-1, \cdots, 1, 0 \end{cases}$$

$$d_k \equiv \frac{\partial b_{k-1}}{\partial \beta} = \begin{cases} 0 & \text{for } k = n+1, n+2 \\ \alpha d_{k+1} + \beta d_{k+2} + b_{k+1} & , k = n, n-1, \cdots, 1 \end{cases}$$

$c_k = d_k$ for $k = 1, 2, \cdots, n+2$

Further define $d_0 = c_0$.

$c_k = d_k$ for $k = 0, 1, 2, \cdots, n+2$

$$\begin{cases} b_0(\alpha_i, \beta_i) + \delta\alpha \frac{\partial b_0}{\partial \alpha}(\alpha_i, \beta_i) + \delta\beta \frac{\partial b_0}{\partial \beta}(\alpha_i, \beta_i) = 0 \\ b_1(\alpha_i, \beta_i) + \delta\alpha \frac{\partial b_1}{\partial \alpha}(\alpha_i, \beta_i) + \delta\beta \frac{\partial b_1}{\partial \beta}(\alpha_i, \beta_i) = 0 \end{cases}$$

$$c_k = \frac{\partial b_k}{\partial \alpha} = d_k = \frac{\partial b_{k-1}}{\partial \beta}$$

Thus $\begin{cases} b_0 + c_0\delta\alpha + c_1\delta\beta = 0 \\ b_1 + c_1\delta\alpha + c_2\delta\beta = 0 \end{cases}$

$$\begin{cases} \delta\alpha = (c_1 b_1 - c_2 b_0)/J \\ \delta\beta = (c_1 b_0 - c_0 b_1)/J \end{cases}$$

$$J = c_0 c_2 - c_1^2$$

P.S. This method fails whenever $J = 0$.

§ Bairstow's method

Step 1: Take one initial guess of real (α_0, β_0)

Step 2: Compute $\{b_k\}_{k=0}^{n+2}$ using (α_0, β_0). Check if $b_0 = 0$ and $b_1 = 0$

Step 3: Compute $\{c_k\}_{k=0}^{n+1}$

$$\begin{cases} c_{n+1} = c_n = 0 \\ c_k = b_{k+1} + \alpha_i c_{k+1} + \beta_i c_{k+2} \quad \text{for } k = n-1, n-2, \cdots, 1, 0 \end{cases}$$

Step 4: Compute $J = c_0 c_2 - c_1^2$ and

$$\begin{cases} \alpha_{i+1} = \alpha_i + (c_1 b_1 - c_2 b_0)/J \\ \beta_{i+1} = \beta_i + (c_1 b_0 - c_0 b_1)/J \end{cases}$$

§ Newton-Raphson Method (quadratic order)

Step 1: Take an initial guess P_0.

Step 2: $P_{n+1} = g(P_n) = P_n - \frac{f(P_n)}{f'(P_n)}$

$$f(P_n) \neq 0$$

$$f(P_n + \delta P) = 0$$

$$f(P_n + \delta P) \approx f(P_n) + \delta P \cdot f'(P_n) = 0$$

$$\delta P = -\frac{f(P_n)}{f'(P_n)}$$