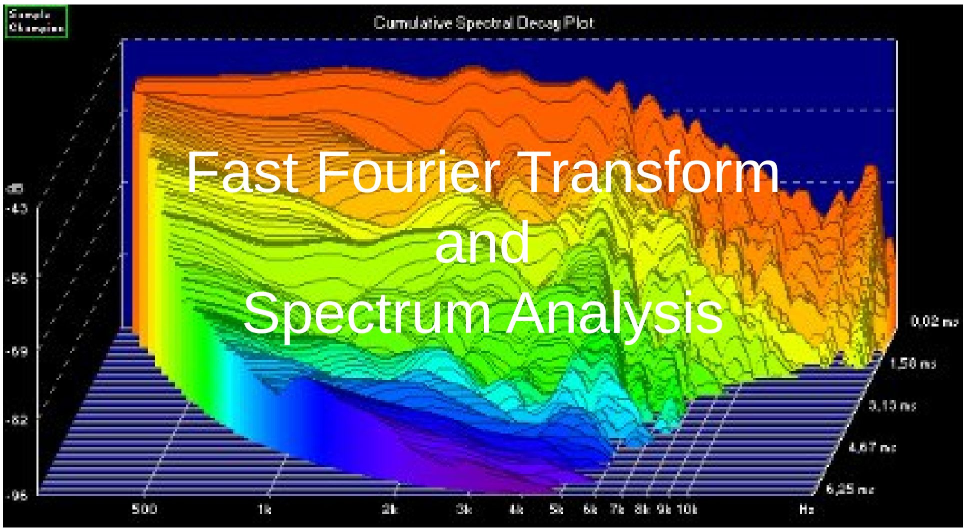
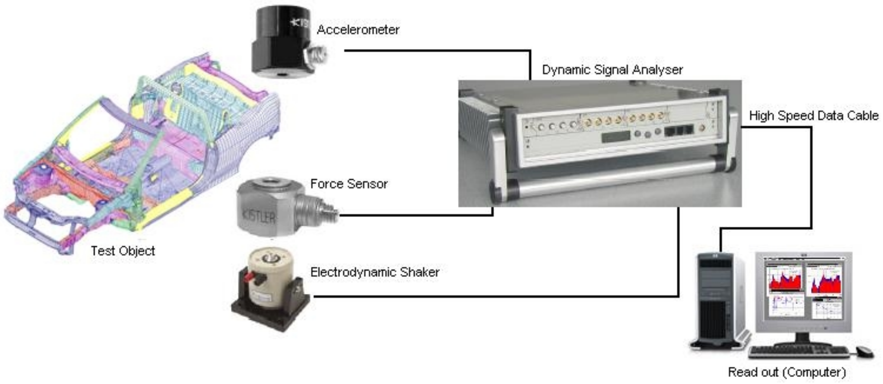
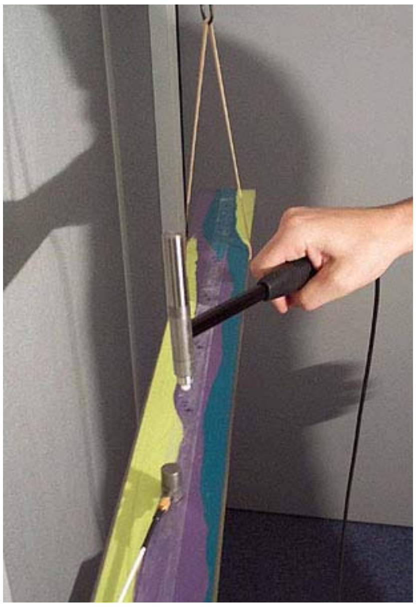
Fast Fourier Transform and Spectrum Analysis



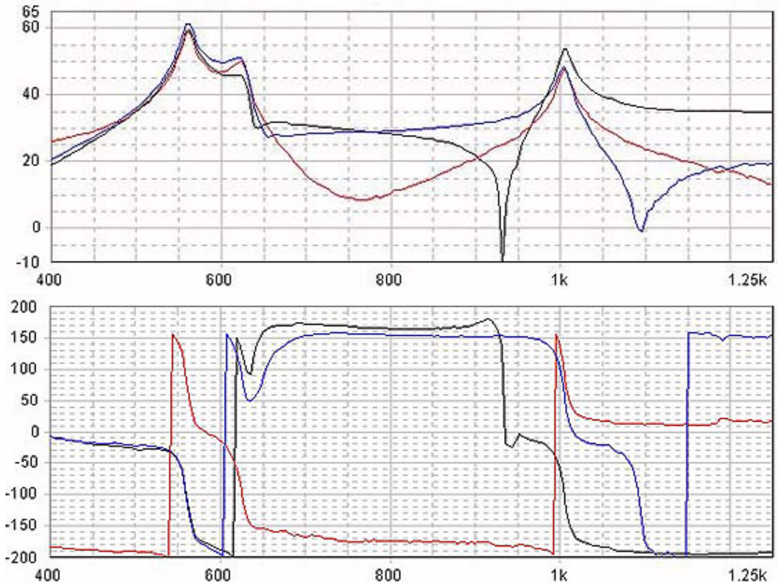
Modal testing



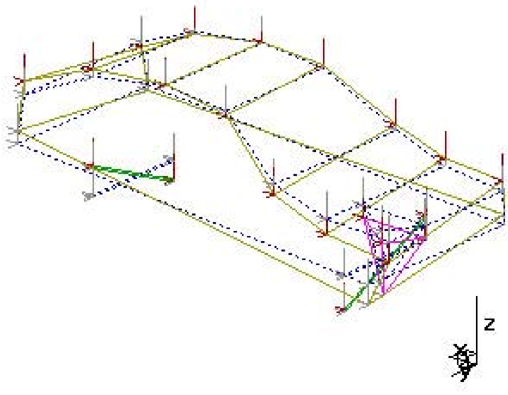
http://en.wikipedia.org/wiki/Modal_testing



Frequency response functions



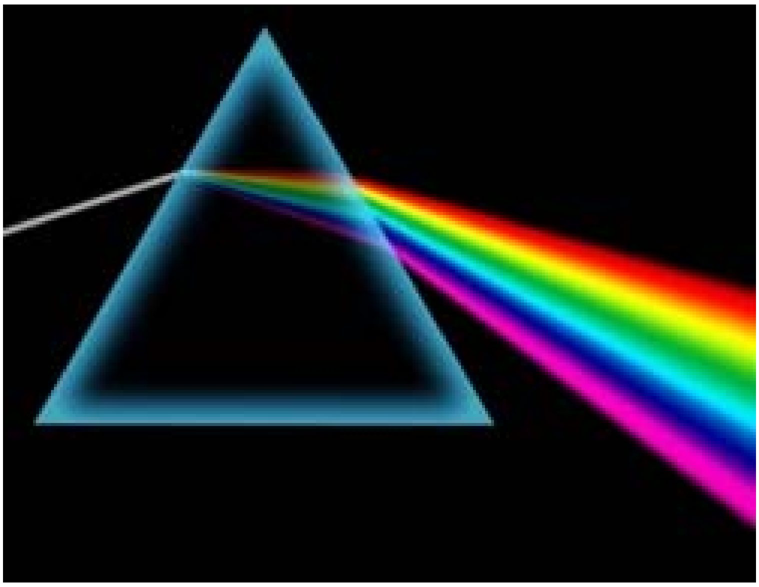
Mode shapes



Application:

http://www.youtube.com/watch?v=9kS3dc3n2Y4

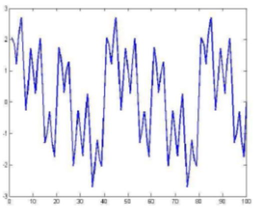
Frequency Spectrum



Time history of a periodic function
--- Fourier series

Joseph Fourier (1768 – 1830)

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos n\omega t + B_n \sin n\omega t\right)$$



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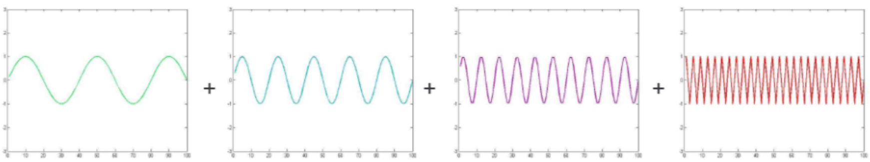
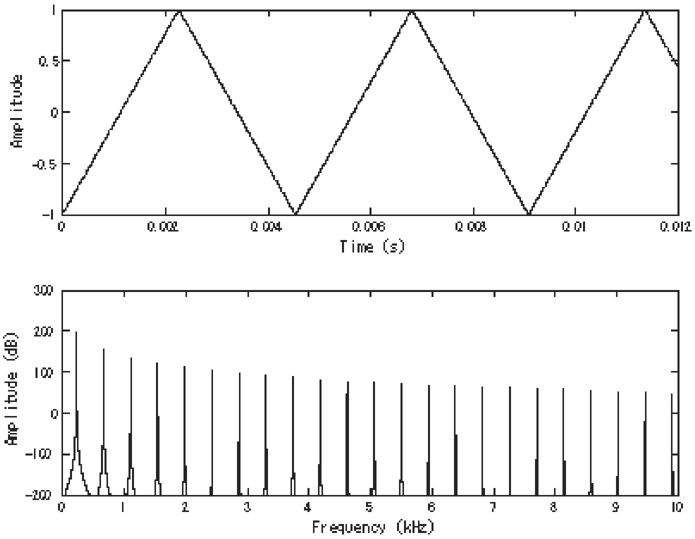
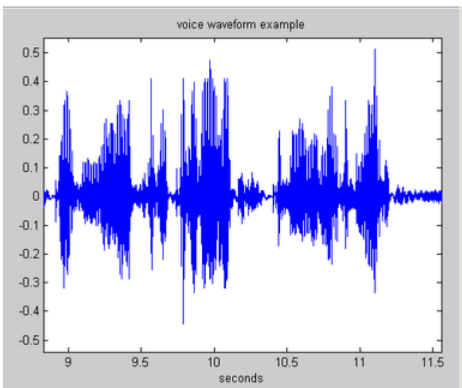
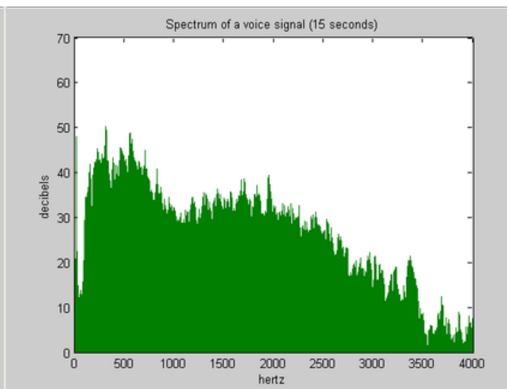


Frequency spectrum



Transient signal
---The Fourier transform





The Fourier transform

Fourier Transform

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

Fourier Synthesis

Fourier Analysis

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

Inverse Fourier Transform

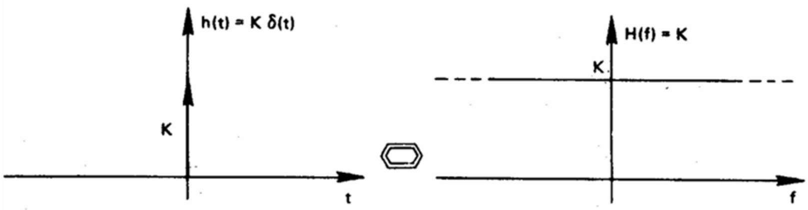


Figure 2-6. Fourier transform of an impulse function.

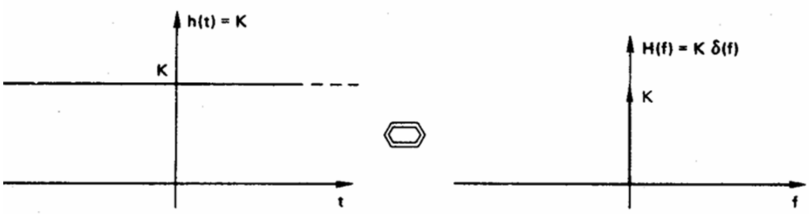


Figure 2-7. Fourier transform of a constant amplitude waveform.

(Ref. E. Oran Brigham, *The Fast Fourier Transform*, Printice-Hall, 1974)

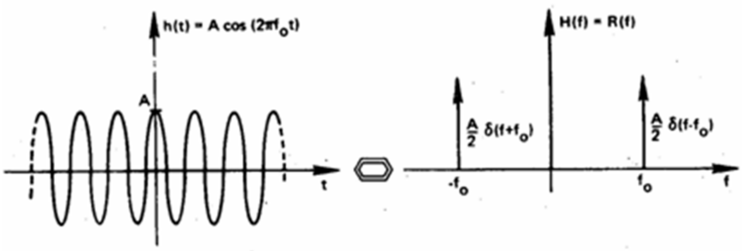


Figure 2-8. Fourier transform of *A* cos (*at*).

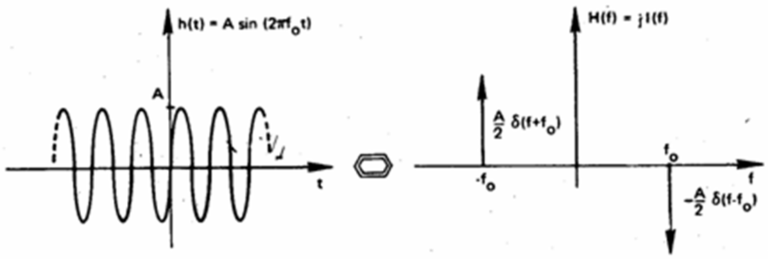


Figure 2-9. Fourier transform of *A* sin (*at*).

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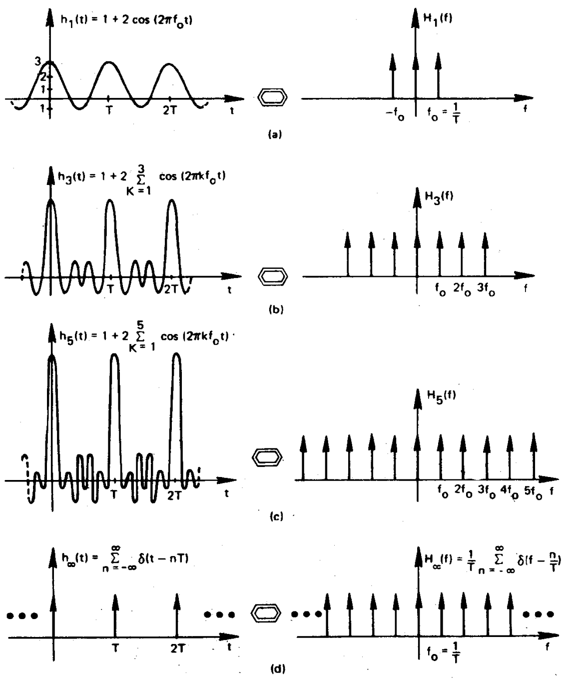

Figure 2-10. Graphical development of the Fourier transform of a sequence of equal distant impulse functions.

$$h(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$\Updownarrow$$

$$H(T) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$$

Fourier transform properties

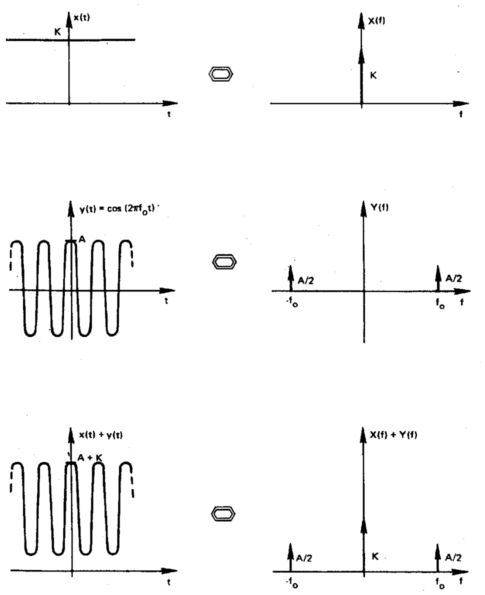

Figure 3-1. The linearity property.

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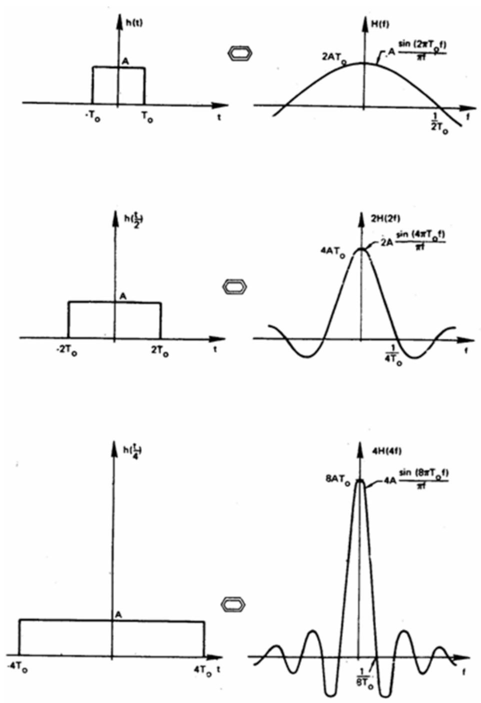

Figure 3-2. Time scaling property.

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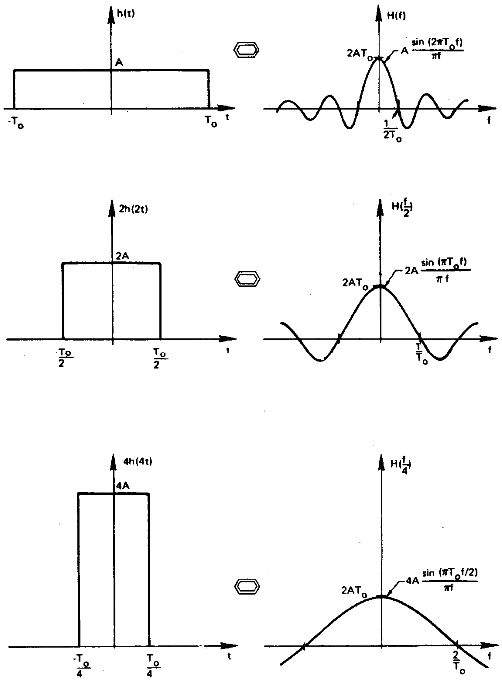

Figure 3-3. Frequency scaling property.

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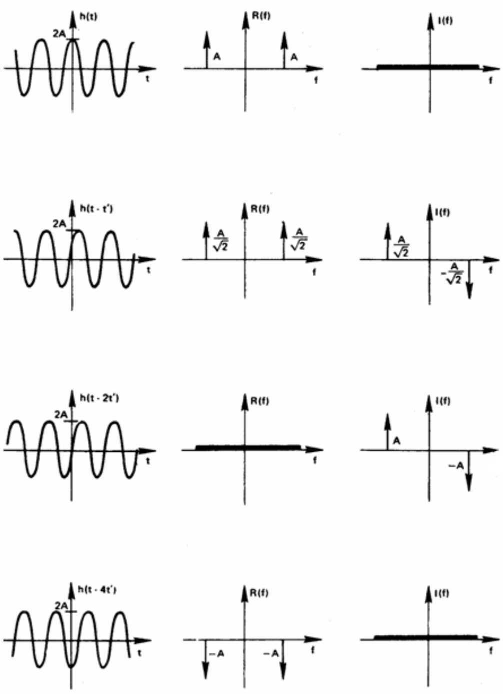

Figure 3-4. Time shifting property.

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Convolution, Correlation, and Power

Convolution --- time domain

$$[g*h](t) \equiv \int_{-\infty}^{+\infty} g(\tau)h(t-\tau)d\tau \quad \Leftrightarrow \quad G(f)H(f)$$

Correlation

$$\left\langle g(\tau)h(\tau+t)\right\rangle \equiv \int_{-\infty}^{+\infty} g(\tau)h(\tau+t)d\tau \quad \Leftrightarrow \quad G(-f)H(f)$$

Autocorrelation if $g = h$

Total power --- Parseval's theorem

$$\int_{-\infty}^{+\infty} |h(t)|^2 \, dt = \int_{-\infty}^{+\infty} |H(f)|^2 \, df$$

Autocorrelation is equal to power spectrum $|G(f)|^2$ in frequency domain.

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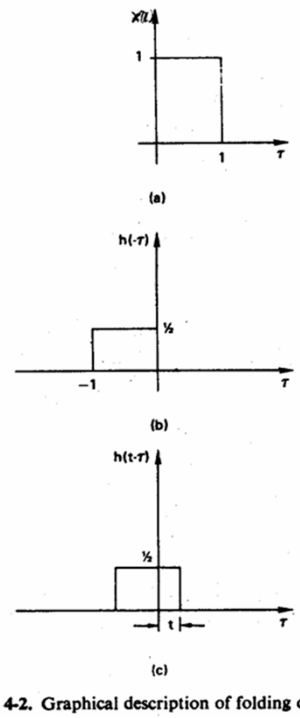

Figure 4-2. Graphical description of folding operation.

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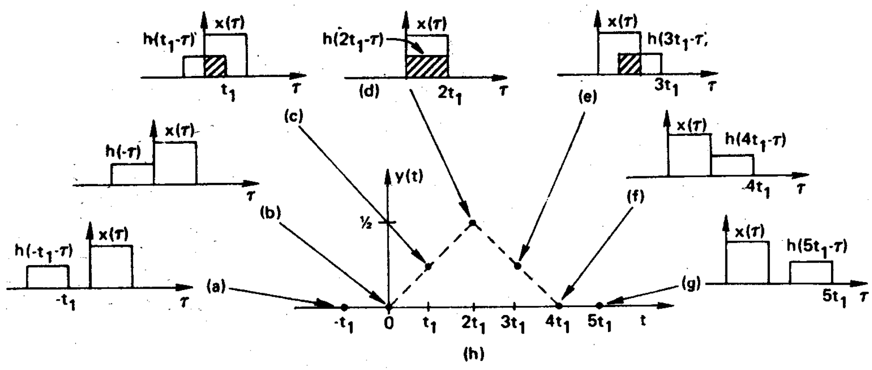

Figure 4-3. Graphical example of convolution.

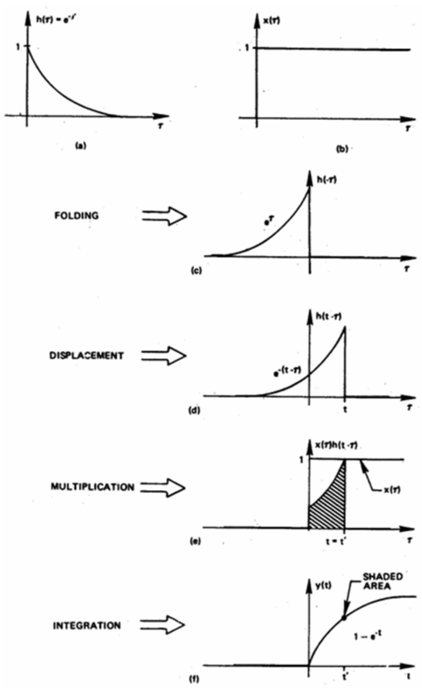

Figure 4-4. Convolution procedure: folding, displacement, multiplication, and integration.

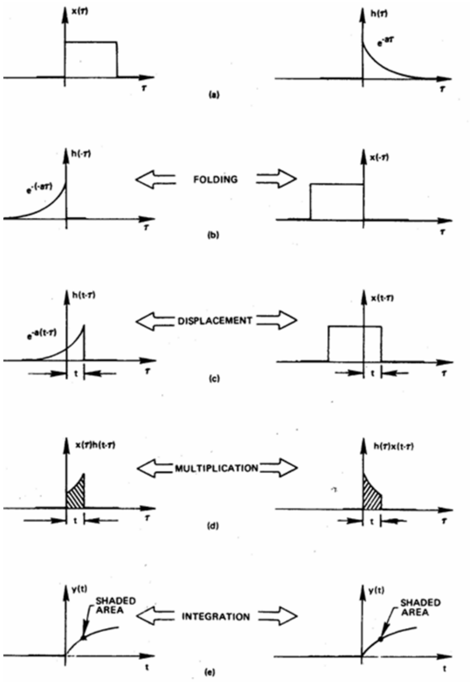

Figure 4-5. Graphical example of convolution by Eqs. (4-1) and (4-3).

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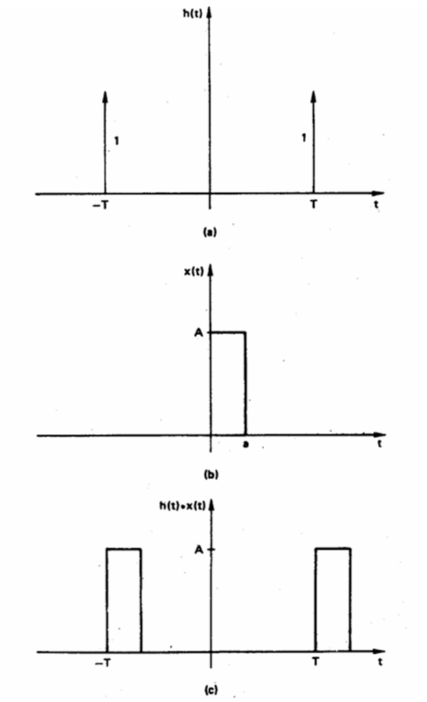

Figure 4-6. Illustration of convolution involving impulse functions.

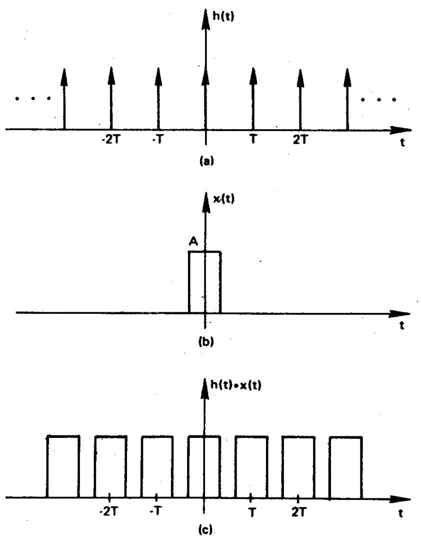

Figure 4-7. Impulse function convolution.

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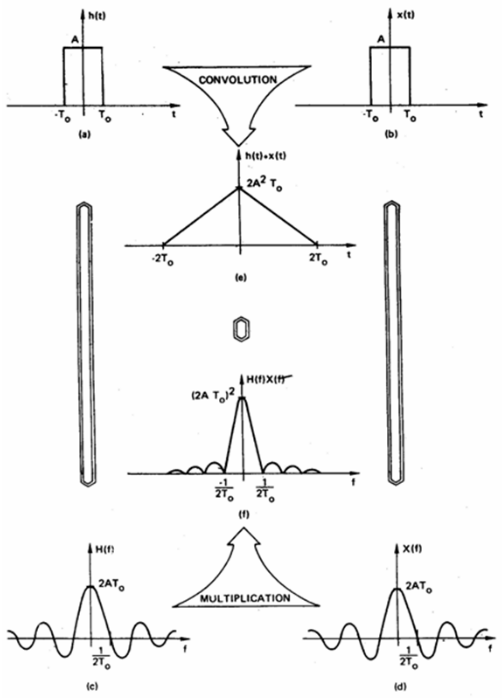

Figure 4-8. Graphical example of the convolution theorem.

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Convolution
--- frequency domain

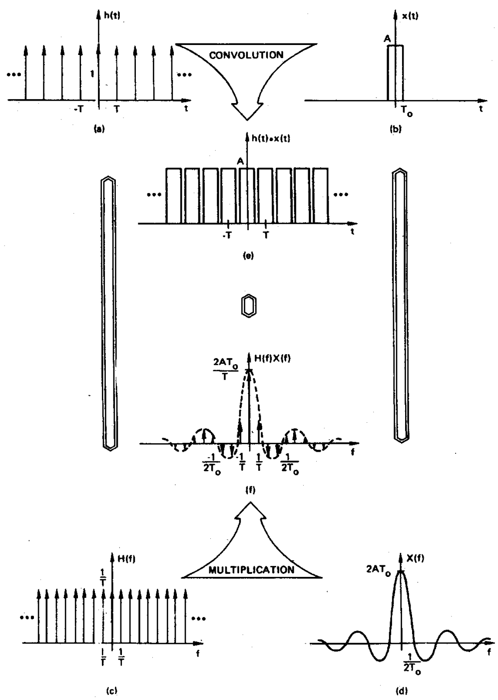

Figure 4-9. Example application of the convolution theorem.

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Fourier series and sampled waveforms

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t)] = \sum_{n=-\infty}^{\infty} \alpha_n e^{j2\pi n f_0 t}$$

$$h(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \qquad \Leftrightarrow \qquad H(T) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$$

$$y(t) = h(t) * x(t) \qquad \Leftrightarrow \qquad Y(f) = X(f)H(f)$$

$$\int_{-\infty}^{\infty} e^{j2\pi ft} df = \delta(t) \qquad \int_{-\infty}^{\infty} e^{j2\pi ft} dt = \delta(f)$$

$$h(t)\delta(t - t_0) = h(t_0)\delta(t - t_0)$$

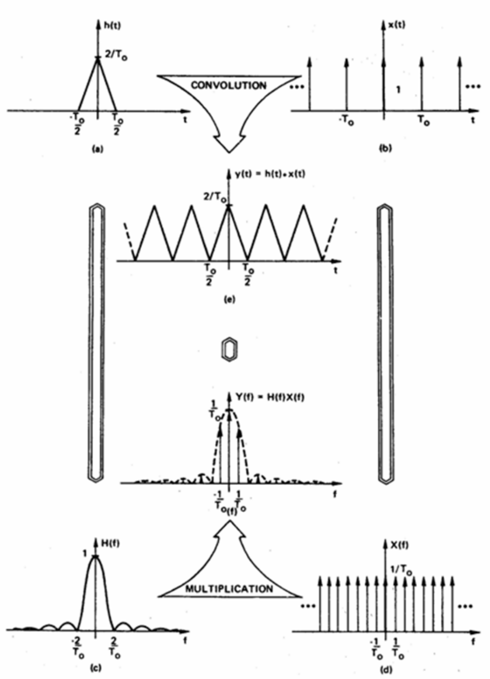

Figure 5-2. Graphical convolution theorem development of the Fourier transform of a periodic triangular waveform.

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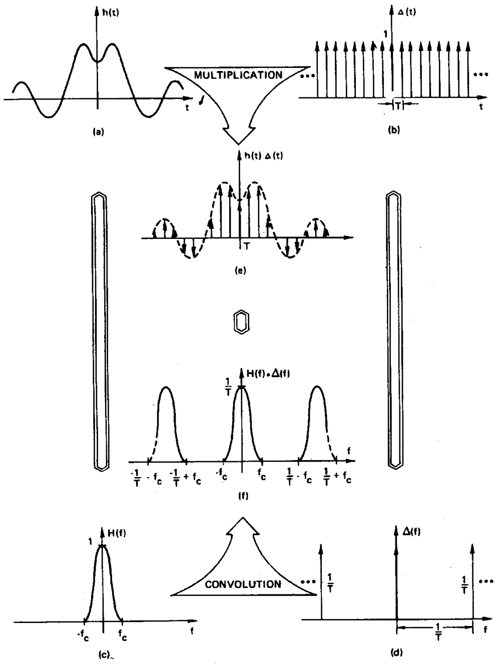

Figure 5-3. Graphical frequency convolution theorem development of the Fourier transform of a sampled waveform.

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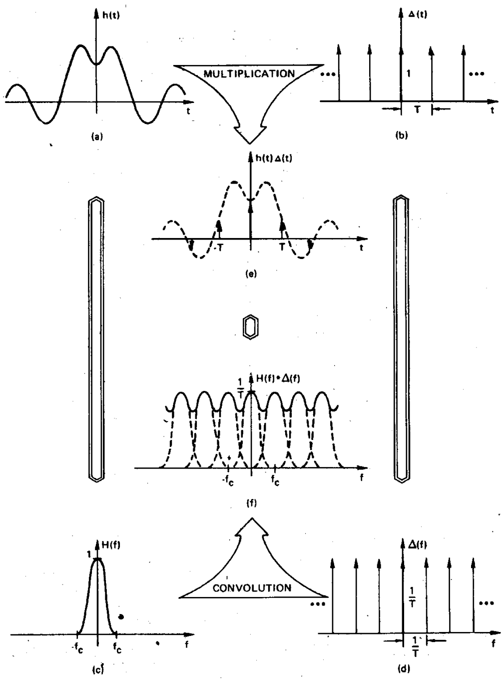

Figure 5-4. Aliased Fourier transform of a waveform sampled at an insufficient rate.

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Sampling theorem

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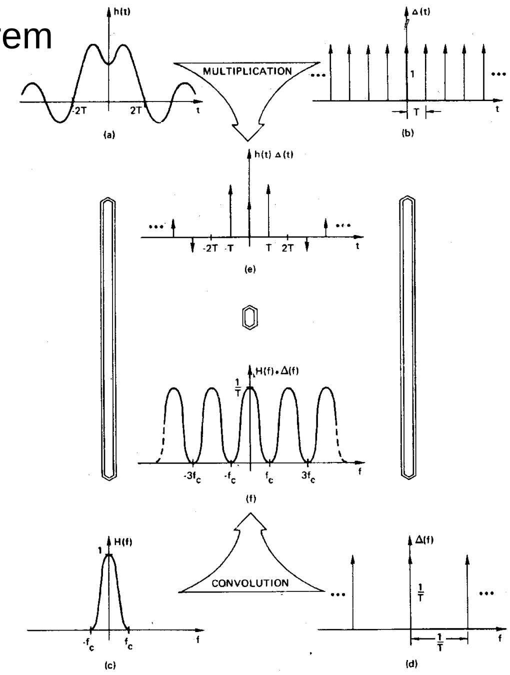

Figure 5-5. Fourier transform of a waveform sampled at the Nyquist sampling rate.

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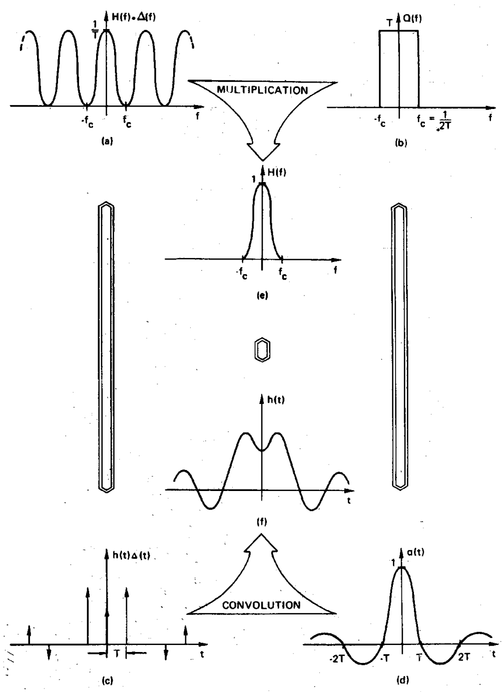

Figure 5-6. Graphical derivation of the sampling theorem.

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The discrete Fourier transform

$$h(t)\Delta_0(t) = h(t)\sum_{k=-\infty}^{\infty}\delta(t-kT) = \sum_{k=-\infty}^{\infty}h(kT)\delta(t-kT)$$

$$h(t)\Delta_0(t)x(t) = \sum_{k=0}^{N-1}h(kT)\delta(t-kT)$$

$$\Delta_1(t) = T_0\sum_{r=-\infty}^{\infty}\delta(t-rT_0)$$

$$[h(t)\Delta_0(t)x(t)]*\Delta_1(t) = \hat{h}(t) = T_0\sum_{r=-\infty}^{\infty}\sum_{k=0}^{N-1}h(kT)\delta(t-kT-rT_0)$$

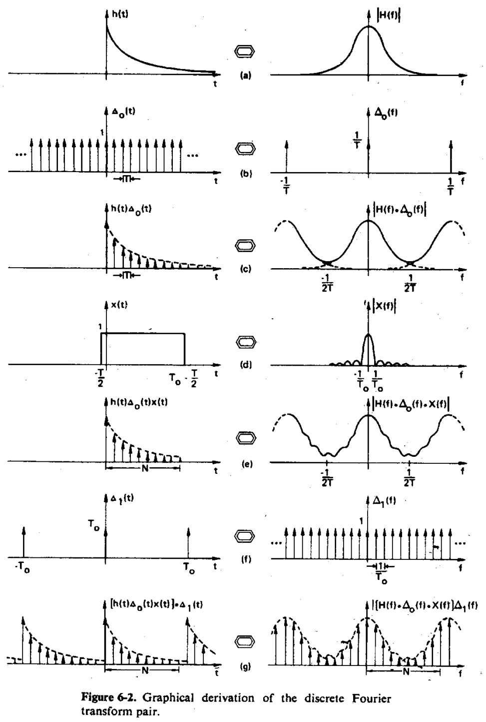

Figure 6-2. Graphical derivation of the discrete Fourier transform pair.

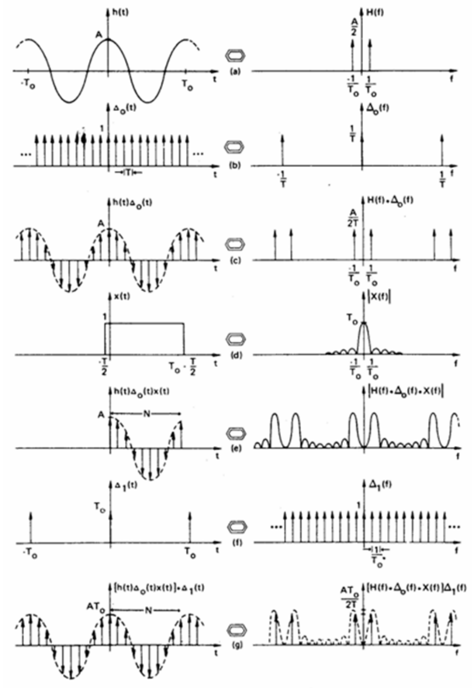

Figure 6-3. Discrete Fourier transform of a band-limited periodic waveform: truncation interval equal to period

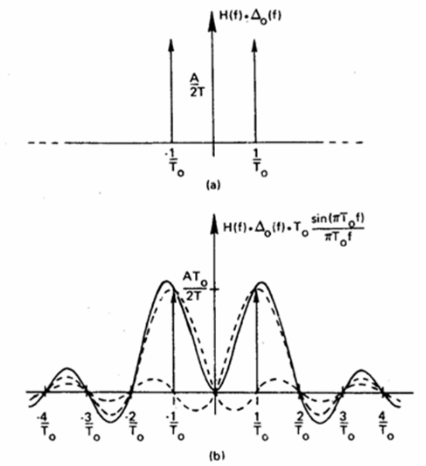

Figure 6-4. Expanded illustration of the convolution of Fig. 6-3(e).

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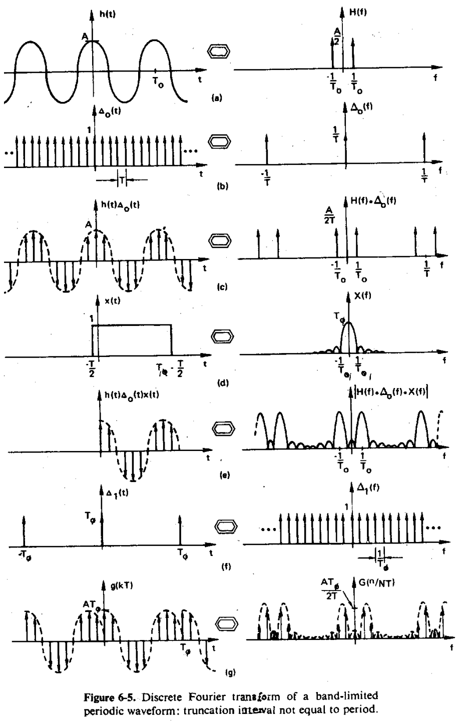

Figure 6-5. Discrete Fourier transform of a band-limited periodic waveform: truncation interval not equal to period.

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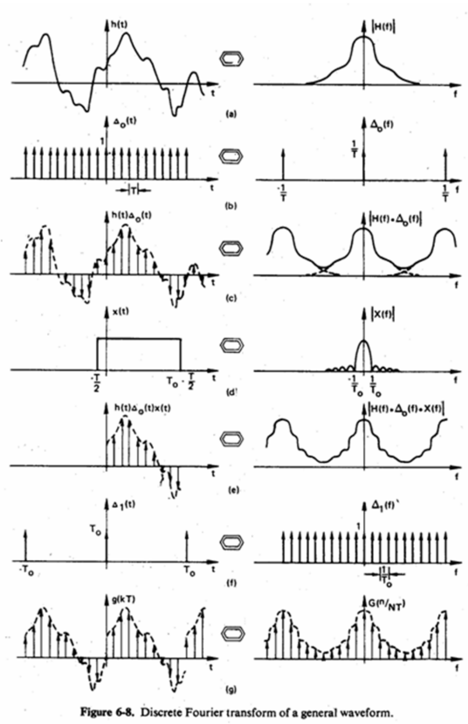
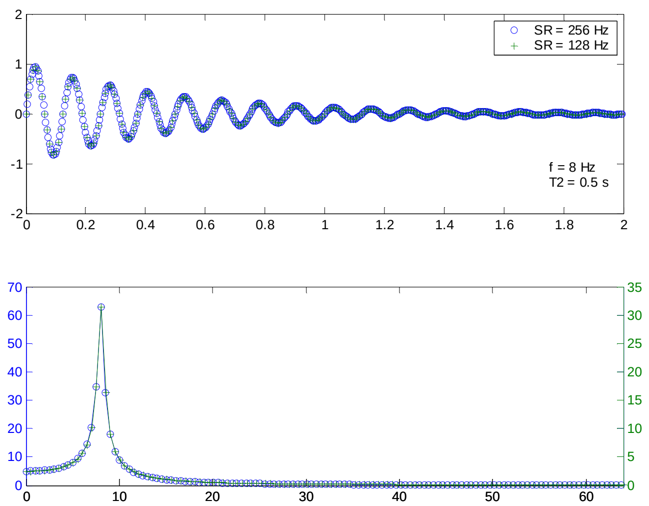
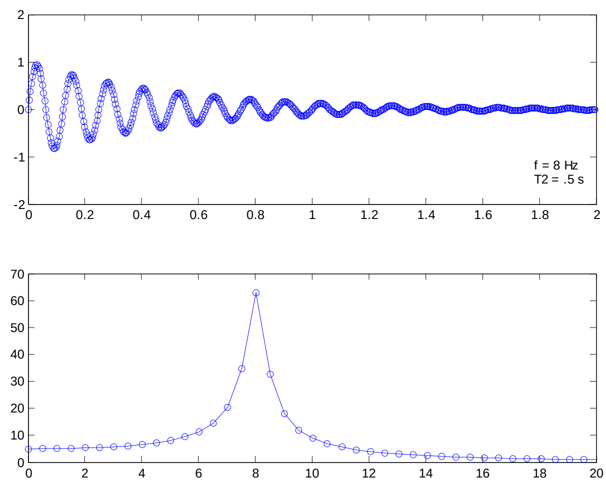


Effect of changing sample rate



- Lowering the sample rate:
  - Reduces the Nyquist frequency, which
  - Reduces the maximum measurable frequency
  - Does not affect the frequency resolution

Effect of changing sampling duration



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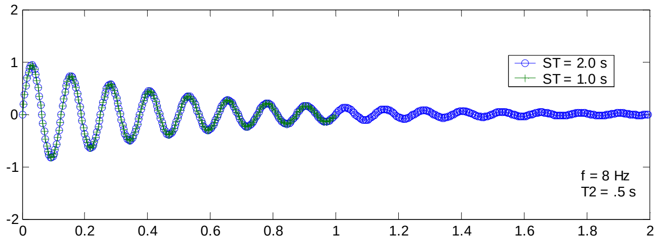

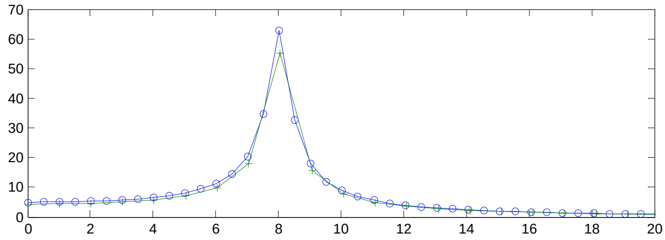

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- Reducing the sampling duration:
  - Lowers the frequency resolution
  - Does not affect the range of frequencies you can measure

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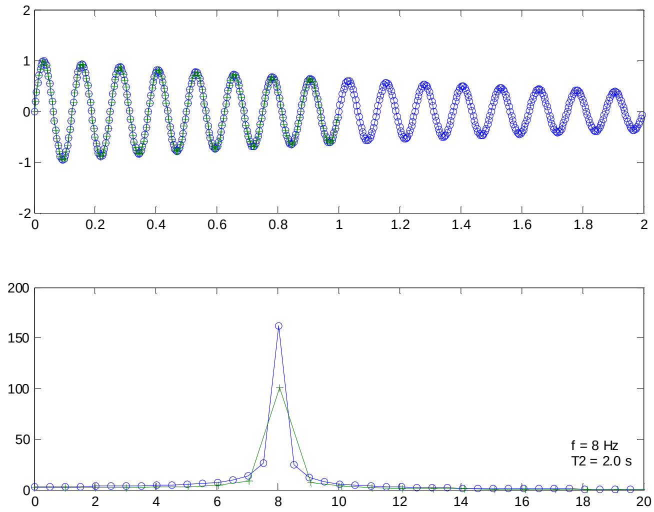

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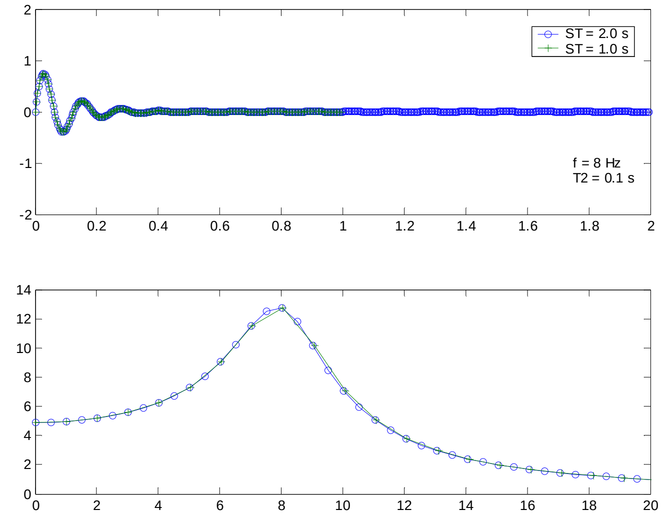

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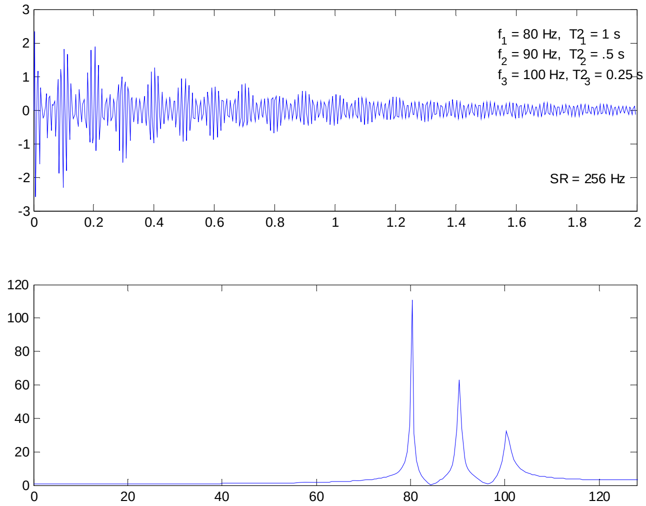
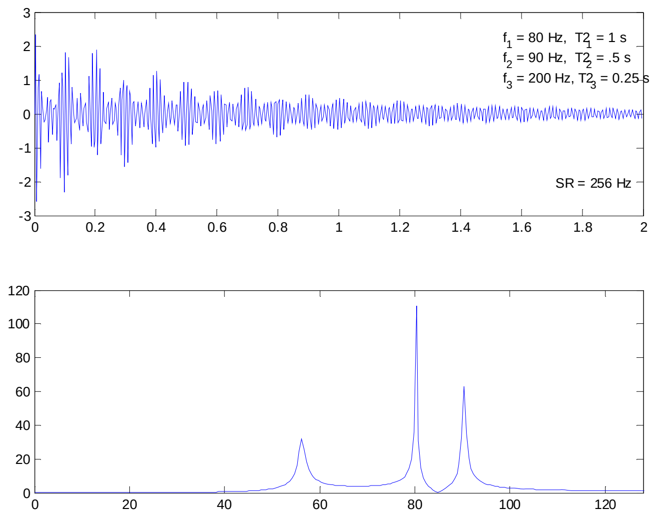
Measuring multiple frequencies





The radix-2 decimation-in-time (DIT) FFT - The Cooley-Tukey algorithm

$$X\left(\frac{n}{NT}\right) = \sum_{k=0}^{N-1} x_0(kT) e^{-j2\pi nk/N} \qquad n = 0,1,\cdots,N-1$$

$$\frac{n}{NT} \rightarrow n; kT \rightarrow k; W = e^{-j2\pi/N}$$

$$X(n) = \sum_{k=0}^{N-1} x_0(k) W^{nk} \qquad n = 0,1,\cdots,N-1$$

$$(1) \mathrm{N} = 2^2 = 4$$

$$k = 0,1,2,3 \rightarrow k = (k_1, k_0) = 00,01,10,11$$

$$n = 0,1,2,3 \rightarrow n = (n_1, n_0) = 00,01,10,11$$

$$X(n_1, n_0) = \sum_{k_0=0}^{1}\sum_{k_1=0}^{1} x_0(k_1, k_0) W^{(2n_1+n_0)(2k_1+k_0)}$$

$$W^{(2n_1+n_0)(2k_1+k_0)} = W^{(2n_1+n_0)2k_1} W^{(2n_1+n_0)k_0}$$

$$= W^{[4n_1k_1]} W^{2n_0k_1} W^{(2n_1+n_0)k_0}$$

$$= W^{2n_0k_1} W^{(2n_1+n_0)k_0}$$

$$W^{4n_1k_1} = [W^4]^{n_1k_1} = [e^{-j2\pi 4/4}]^{n_1k_1} = 1^{n_1k_1} = 1$$

$$\Rightarrow X(n_1, n_0) = \sum_{k_0=0}^{1} \left[\sum_{k_1=0}^{1} x_0(k_1, k_0) W^{2n_0k_1}\right] W^{(2n_1+n_0)k_0}$$

$$x_1(n_0, k_0) = \sum_{k_1=0}^{1} x_0(k_1, k_0) W^{2n_0k_1}$$

$$x_1(0,0) = x_0(0,0) + x_0(1,0)W^0$$

$$x_1(0,1) = x_0(0,1) + x_0(1,1)W^0$$

$$x_1(1,0) = x_0(0,0) + x_0(1,0)W^2$$

$$x_1(1,1) = x_0(0,1) + x_0(1,1)W^2$$

$$\begin{bmatrix} x_1(0,0) \\ x_1(0,1) \\ x_1(1,0) \\ x_1(1,1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & W^0 & 0 \\ 0 & 1 & 0 & W^0 \\ 1 & 0 & W^2 & 0 \\ 0 & 1 & 0 & W^2 \end{bmatrix} \begin{bmatrix} x_0(0,0) \\ x_0(0,1) \\ x_0(1,0) \\ x_0(1,1) \end{bmatrix}$$

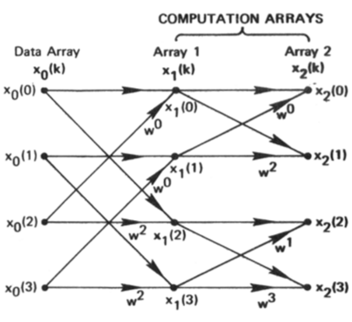
$$x_2(n_0, n_1) = \sum_{k_0=0}^{1} x_1(n_0, k_0) W^{(2n_1+n_0)k_0}$$

$$\begin{bmatrix} x_2(0,0) \\ x_2(0,1) \\ x_2(1,0) \\ x_2(1,1) \end{bmatrix} = \begin{bmatrix} 1 & W^0 & 0 & 0 \\ 1 & W^2 & 0 & 0 \\ 0 & 0 & 1 & W^1 \\ 0 & 0 & 1 & W^3 \end{bmatrix} \begin{bmatrix} x_1(0,0) \\ x_1(0,1) \\ x_1(1,0) \\ x_1(1,1) \end{bmatrix}$$

$$x_2(n_0, n_1) \Rightarrow X(n_1, n_0)$$

$$X(n_1, n_0) = \sum_{k_0=0}^{1} [\sum_{k_1=0}^{1} x_0(k_1, k_0) W^{2n_0k_1}] W^{(2n_1+n_0)k_0}$$

$$\begin{bmatrix} X(0,0) \\ X(1,0) \\ X(0,1) \\ X(1,1) \end{bmatrix} = \begin{bmatrix} 1 & W^0 & 0 & 0 \\ 1 & W^2 & 0 & 0 \\ 0 & 0 & 1 & W^1 \\ 0 & 0 & 1 & W^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & W^0 & 0 \\ 0 & 1 & 0 & W^0 \\ 1 & 0 & W^2 & 0 \\ 0 & 1 & 0 & W^2 \end{bmatrix} \begin{bmatrix} x_0(0,0) \\ x_0(0,1) \\ x_0(1,0) \\ x_0(1,1) \end{bmatrix}$$



$$(2)N = 8 = 2^3$$

$$X(n) = \sum_{k=0}^{N-1} x_0(k) W^{nk}$$

$$\rightarrow X(n_2, n_1, n_0) = \sum_{k_0=0}^{1} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} x_0(k_2, k_1, k_0) W^{(4n_2+2n_1+n_0)(4k_2+2k_1+k_0)}$$

$$W^{(4n_2+2n_1+n_0)(4k_2+2k_1+k_0)}$$

$$= W^{(4n_2+2n_1+n_0)4k_2} W^{(4n_2+2n_1+n_0)2k_1} W^{(4n_2+2n_1+n_0)k_0}$$

$$\Rightarrow W^{(4n_2+2n_1+n_0)4k_2} = [W^{8(n_1k_2)}][W^{8(n_2k_2)}]W^{4n_0k_2} = W^{4n_0k_2}$$

$$\Rightarrow W^{(4n_2+2n_1+n_0)2k_1} = [W^{8(n_2k_1)}]W^{(2n_1+n_0)2k_1} = W^{(2n_1+n_0)2k_1}$$

$$W^8 = [e^{-j2\pi/8}]^8 = 1$$

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$$X(n_2, n_1, n_0) = \sum_{k_0=0}^{1} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} x_0(k_2, k_1, k_0) W^{(4n_2+2n_1+n_0)(4k_2+2k_1+k_0)}$$

$$= \sum_{k_0=0}^{1} [\sum_{k_1=0}^{1} [\sum_{k_2=0}^{1} x_0(k_2, k_1, k_0) W^{4n_0k_2}] W^{(2n_1+n_0)2k_1}] W^{(4n_2+2n_1+n_0)k_0}$$

$$x_1(n_0, k_1, k_0) = \sum_{k_2=0}^{1} x_0(k_2, k_1, k_0) W^{4n_0k_2}$$

$$x_2(n_0, n_1, k_0) = \sum_{k_2=0}^{1} x_1(n_0, k_1, k_0) W^{(2n_1+n_0)2k_1}$$

$$x_3(n_0, n_1, n_2) = \sum_{k_2=0}^{1} x_2(n_0, n_1, k_0) W^{(4n_2+2n_1+n_0)k_0}$$

$$x_3(n_0, n_1, n_2) \Rightarrow X(n_2, n_1, n_0)$$

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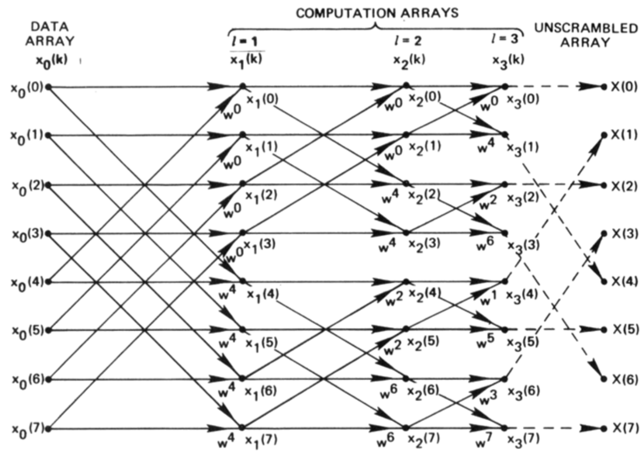
N=8



$$(3)N = 2^{\gamma}$$

$$X(n) = \sum_{k=0}^{N-1} x_0(k) W^{nk}$$

$$\Rightarrow X(n_{\gamma-1}, n_{\gamma-2}, \cdots, n_0) = \sum_{k_0=0}^{1} \sum_{k_1=0}^{1} \cdots \sum_{k_{\gamma-1}=0}^{1} x_0(k_{\gamma-1}, k_{\gamma-2}, \cdots, k_0) W^{nk}$$

$$n = 2^{\gamma-1} n_{\gamma-1} + 2^{\gamma-2} n_{\gamma-2} + \cdots + n_0$$

$$k = 2^{\gamma-1} k_{\gamma-1} + 2^{\gamma-2} k_{\gamma-2} + \cdots + k_0$$

$$W^{nk} = W^{(2^{\gamma-1} n_{\gamma-1} + 2^{\gamma-2} n_{\gamma-2} + \cdots + n_0)(2^{\gamma-1} k_{\gamma-1} + 2^{\gamma-2} k_{\gamma-2} + \cdots + k_0)}$$

$$= W^{(2^{\gamma-1} n_{\gamma-1} + 2^{\gamma-2} n_{\gamma-2} + \cdots + n_0) 2^{\gamma-1} k_{\gamma-1}}$$

$$\times W^{(2^{\gamma-1} n_{\gamma-1} + 2^{\gamma-2} n_{\gamma-2} + \cdots + n_0) 2^{\gamma-2} k_{\gamma-2}}$$

$$\times \cdots W^{(2^{\gamma-1} n_{\gamma-1} + 2^{\gamma-2} n_{\gamma-2} + \cdots + n_0) k_0}$$

$$W^{(2^{\gamma-1}n_{\gamma-1}+2^{\gamma-2}n_{\gamma-2}+\cdots+n_0)2^{\gamma-1}k_{\gamma-1}}$$
$$=[W^{2^{\gamma}(2^{\gamma-2}n_{\gamma-1}k_{\gamma-1})}][W^{2^{\gamma}(2^{\gamma-3}n_{\gamma-2}k_{\gamma-1})}]\cdots[W^{2^{\gamma}(n_1k_{\gamma-1})}]W^{2^{\gamma-1}(n_0k_{\gamma-1})}$$
$$=W^{2^{\gamma-1}(n_0k_{\gamma-1})}$$
$$W^{2^{\gamma}}=W^N=[e^{-j2\pi/N}]^N=1$$

$$W^{(2^{\gamma-1}n_{\gamma-1}+2^{\gamma-2}n_{\gamma-2}+\cdots+n_0)2^{\gamma-2}k_{\gamma-2}}$$
$$=[W^{2^{\gamma}(2^{\gamma-3}n_{\gamma-1}k_{\gamma-2})}][W^{2^{\gamma}(2^{\gamma-4}n_{\gamma-2}k_{\gamma-2})}]\cdots W^{2^{\gamma-1}(n_1k_{\gamma-2})}W^{2^{\gamma-2}(n_0k_{\gamma-2})}$$
$$=W^{2^{\gamma-1}(n_1k_{\gamma-2})}W^{2^{\gamma-2}(n_0k_{\gamma-2})}$$
$$=W^{(2n_1+n_0)2^{\gamma-2}k_{\gamma-2}}$$

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$$X(n_{\gamma-1},n_{\gamma-2},\cdots,n_0)=\sum_{k_0=0}^{1}\sum_{k_1=0}^{1}\cdots\sum_{k_{\gamma-1}=0}^{1}x_0(k_{\gamma-1},k_{\gamma-2},\cdots,k_0)$$
$$\times W^{2^{\gamma-1}(n_0k_{\gamma-1})}W^{(2n_1+n_0)2^{\gamma-2}k_{\gamma-2}}\cdots W^{(2^{\gamma-1}n_{\gamma-1}+2^{\gamma-2}n_{\gamma-2}+\cdots+n_0)k_0}$$

$$x_1(n_0,k_{\gamma-2},\cdots,k_0)=\sum_{k_{\gamma-1}=0}^{1}x_0(k_{\gamma-1},k_{\gamma-2},\cdots,k_0)W^{2^{\gamma-1}(n_0k_{\gamma-1})}$$
$$x_2(n_0,n_1,\cdots,k_0)=\sum_{k_{\gamma-2}=0}^{1}x_1(n_0,k_{\gamma-2},\cdots,k_0)W^{(2n_1+n_0)2^{\gamma-2}k_{\gamma-2}}$$
$$\vdots$$
$$x_\gamma(n_0,n_1,\cdots,n_{\gamma-1})=\sum_{k_0=0}^{1}x_{\gamma-1}(n_0,n_1,\cdots,n_{\gamma-1})W^{(2^{\gamma-1}n_{\gamma-1}+2^{\gamma-2}n_{\gamma-2}+\cdots+n_0)k_0}$$
$$x_\gamma(n_0,n_1,\cdots,n_{\gamma-1})\Rightarrow X(n_{\gamma-1},n_{\gamma-2},\cdots,n_0)$$

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直接計算:

$$X(n)=\sum_{k=0}^{N-1}x_0(k)W^{nk}$$

$$n=0,1,\cdots,N-1$$

$\Rightarrow N^2$次複數乘法

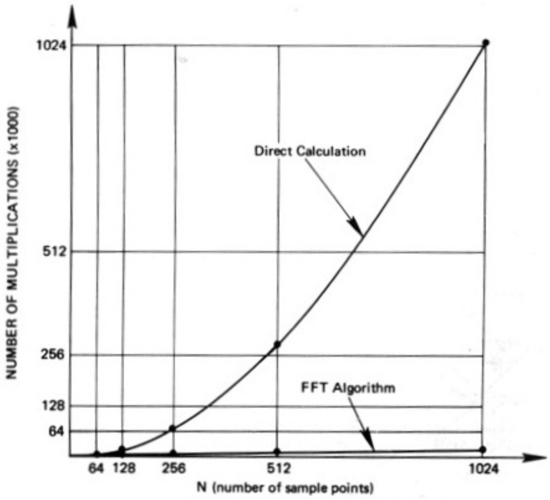
基底2之FFT算法:

有γ個方程式,

每方程式，有N個乘法

$\Rightarrow N\cdot\gamma$次複數乘法

$$\frac{N\cdot\gamma}{N^2}=\frac{\gamma}{N}$$



$$h(t)=\int_{-\infty}^{\infty}H(f)e^{j2\pi ft}df$$

$$=\int_{-\infty}^{\infty}[R(f)+jI(f)][\cos(2\pi ft)+j\sin(2\pi ft)]df$$

$$=\int_{-\infty}^{\infty}[R(f)\cos(2\pi ft)-I(f)\sin(2\pi ft)]df$$

$$+j\int_{-\infty}^{\infty}[R(f)\sin(2\pi ft)+I(f)\cos(2\pi ft)]df$$

$$=\left[\int_{-\infty}^{\infty}[R(f)\cos(2\pi ft)-I(f)\sin(2\pi ft)]df\right.$$

$$\left.-j\int_{-\infty}^{\infty}[R(f)\sin(2\pi ft)+I(f)\cos(2\pi ft)]df\right]^*$$

$$=\left[\int_{-\infty}^{\infty}R(f)e^{-j2\pi ft}df-j\int_{-\infty}^{\infty}I(f)e^{-j2\pi ft}df\right]^*$$

$$=\left[\int_{-\infty}^{\infty}H^*(f)e^{-j2\pi ft}df\right]^*$$